NAVAL SHIP RESEARCH AND DEVELOPMENT CENTER
Bethesda, Maryland 20034

THE CONVERGING FACTORS FOR THE SINE AND COSINE INTEGRALS

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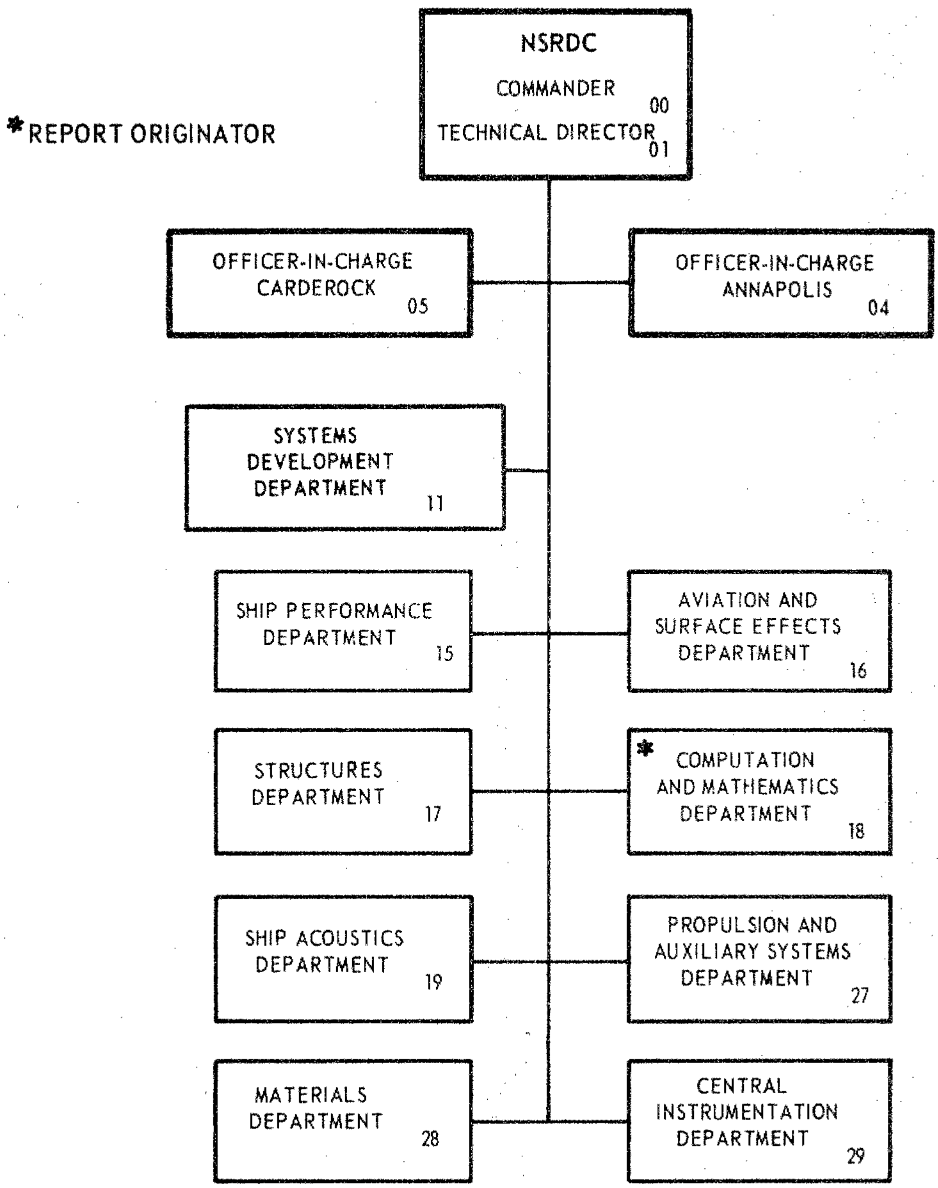

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By

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ABSTRACT

The theory of the converging factors for the sine and cosine integrals is developed and then applied to the calculation on a CDC 6700 system of tables of these factors and their reduced derivatives to about 35 decimal places. The factors are then used in conjunction with appropriately truncated asymptotic series to produce tables of si(x), Si(x), and Ci(x) to 28 decimals by a computer program, for successive integer values of x ranging from 1 to 70, inclusive. Interpolation in the tables by Taylor series is illustrated by numerical examples, which include a new, extended calculation of the Gibbs constant.

ADMINISTRATIVE INFORMATION

Work on this research was authorized by the Naval Ship Systems Command under the Mathematical Sciences Program. Necessary funds were allocated under Subproject SR 0140301, Task 15324.

INTRODUCTION

The sine and cosine integrals are known to be simply related to the exponential integral[1,2]. This fact has motivated this study, which derives from a detailed examination by Murnaghan and Wrench[3] of the converging factor for the exponential integral.

The first tables of numerical values of the sine and cosine integrals appear to be those of Bretschneider[4], which first appeared in 1843. These consisted of 20-decimal approximations corresponding to the first 10 integer arguments. In 1870 Glaisher[5] published more voluminous tables to 11 and 18 decimal places. The British Association for the Advancement of Science[6] has published such tables to 10 and 11 places. The most extensive tabulation has been that of the National Bureau of Standards[7,8] (1940 and 1942) to 9 and 10 decimals (15 decimals near the extrema), occupying three volumes.

In addition to these tables, a large number of others to less precision (ranging from 3 to 7 decimals) have been published. These are listed (along with those cited) by A. Fletcher[9] and his associates.

In this report methods are developed for the evaluation of the converging factors for the sine and cosine integrals to high precision, and these procedures are exemplified by the calculation of appended tables of these factors and their reduced derivatives to 33 and 35 decimal places. Application of these tables is illustrated by specific examples.

These algorithms have been programmed for the CDC 6700 system and were used to calculate 28-place tables of si(x), Si(x), and Ci(x) for x = 1(1)70, which are also appended to this report (Tables 7-9).

[1] A complete list of references is given on pages 98-99.

THE SINE AND COSINE INTEGRALS

The sine and cosine integrals are generally defined by the relations

$$\mathrm{Si}(z) = \int_0^z \frac{\sin t}{t}\, dt \,, \tag{1}$$

$$\mathrm{Ci}(z) = \int_\infty^z \frac{\cos t}{t}\, dt \quad (|\arg z| < \pi) \tag{2}$$

These integrals possess the power-series developments

$$\mathrm{Si}(z) = z - \frac{z^3}{3 \cdot 3!} + \frac{z^5}{5 \cdot 5!} - \cdots \,, \tag{3}$$

$$\mathrm{Ci}(z) = \gamma + \ln z - \frac{z^2}{2 \cdot 2!} + \frac{z^4}{4 \cdot 4!} - \cdots \,, \tag{4}$$

which are convergent for all values of z (excluding zero in the second expansion). However, for argument z of absolute value exceeding 10, say, it is preferable to evaluate these functions by asymptotic series, especially when modified by an appropriate converging factor, which is the subject of this report.

Instead of Si(z) we shall here consider the complementary integral

$$\mathrm{si}(z) = \int_\infty^z \frac{\sin t}{t}\, dt = \mathrm{Si}(z) - \frac{\pi}{2} \tag{5}$$

Then, if we use the Euler relations

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \,, \tag{6}$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \,, \tag{7}$$

we can write

$$si(z) = \frac{1}{2i}\{ Ei(iz) - Ei(-iz) \} = -Im\ Ei(-iz) , \tag{8}$$

$$Ci(z) = \frac{1}{2}\{Ei(iz) + Ei(-iz) \} = Re\ Ei(-iz) , \tag{9}$$

where $Ei(-z)$ is the exponential integral, defined by

$$Ei(-z) = \int_{\infty}^{z} \frac{e^{-t}}{t}\, dt . \tag{10}$$

THE ASYMPTOTIC SERIES FOR THE SINE AND COSINE INTEGRALS AND THEIR CONVERGING FACTORS

In a previous report[3] F. D. Murnaghan and the present writer developed the series

$$-Ei(-z) = \frac{e^{-z}}{z}\{1 - \frac{1}{z} + \frac{2!}{z^2} - \cdots + (-1)^{n-1}\frac{(n-1)!}{z^{n-1}} + (-1)^n \frac{n!}{z^n}\Gamma_n(z)\}, \tag{11}$$

where $\Gamma_n(z)$ is the appropriate converging factor, which is given by the integral

$$\Gamma_n(z) = \frac{1}{n!}\int_0^{\infty} \frac{e^{-u} u^n}{1+\frac{u}{z}}\, du , \quad n = 1, 2, 3, \cdots , \tag{12}$$

and which satisfies both the recurrence relation

$$\Gamma_n(z) = 1 - \frac{n+1}{z}\Gamma_{n+1}(z) \tag{13}$$

and the differential equation

$$\frac{d}{dz} \Gamma_n(z) = \left(\frac{n+1}{z} + 1\right) \Gamma_n(z) - 1 \quad . \tag{14}$$

From the asymptotic series for Ei(-z) we then deduce, upon replacing z by iz, the series

$$si(z) = -z^{-1}[C(z) \cos z + D(z) \sin z] , \tag{15}$$

$$Ci(z) = z^{-1}[C(z) \sin z - D(z) \cos z] , \tag{16}$$

where

$$C(z) = 1 - \frac{2!}{z^2} + \frac{4!}{z^4} - \cdots + \frac{(-1)^k (2k)!}{z^{2k}} \operatorname{Re} \Gamma_{2k}(-iz) , \tag{17}$$

$$D(z) = \frac{1}{z} - \frac{3!}{z^3} + \frac{5!}{z^5} - \cdots + \frac{(-1)^k (2k+1)!}{z^{2k+1}} \operatorname{Re} \Gamma_{2k+1}(-iz) , \tag{18}$$

This is equivalent to the corresponding series given by Dingle[10].

If we set

$$\Pi_n(z) = \operatorname{Re} \Gamma_n(-iz) \tag{19}$$

and

$$\Omega_n(z) = -\operatorname{Im} \Gamma_n(-iz) , \tag{20}$$

so that $\Gamma_n(-iz) = \Pi_n(z) - i\,\Omega_n(z)$, then we deduce from Equation (12) the representations

$$\Pi_n(z) = \frac{z^2}{n!} \int_0^\infty \frac{e^{-u} u^n}{u^2 + z^2} \, du \tag{21}$$

$$\Omega_n(z) = \frac{z}{n!}\int_0^{\infty} \frac{e^{-u}\,u^{n+1}}{u^2+z^2}\,du = \frac{n+1}{z}\,\Pi_{n+1}(z) \tag{22}$$

for the components of $\Gamma_n(-iz)$ as real improper integrals.

In the previously cited report of Murnaghan and Wrench[3] the following series was developed for $\Gamma_n(z)$:

$$\Gamma_n(z) = a_0(\beta) + \sum_{j=1}^{\infty} \frac{a_j(\beta)}{z^j} \tag{23}$$

where $\beta = 1 - \frac{n+1}{z}$, $a_0(\beta) = \frac{1}{2-\beta}$, $a_j(\beta) = \frac{(1-\beta)P_{j-1}(1-\beta)}{(2-\beta)^{2j+1}}$. Here

$P_j(v)$ is a polynomial in v of degree j which satisfies the recurrence

$$P_{j+1}(v) = \{(j+1)v-(j+2)\}\,P_j(v) - v(v+1)\,\frac{d}{dv}\,P_j(v) \quad , \tag{24}$$

and $P_0(v) = 1$.

If we replace $n+1$ by s and z by $-is$, then Equation (23) becomes

$$\Gamma_{s-1}(-is) = \frac{1}{1+i} + i\sum_{j=1}^{\infty} \frac{P_{j-1}(i)}{(1+i)^{2j+1}(-is)^j} \tag{25}$$

Since $(1+i)^2 = 2i$ and $\frac{i}{1+i} = \frac{1}{2}(1+i)$, we have

$$\Gamma_{s-1}(-is) = \frac{1}{2}(1-i) + \frac{1}{2}\sum_{j=1}^{\infty} \frac{(i+i)\,P_{j-1}(i)}{(2s)^j} \tag{26}$$

Now, if we set $n = s-1$ and $z = -is$ in Equation (13) we obtain

$$\Gamma_s(-is) = -i + i\,\Gamma_{s-1}(-is) \tag{27}$$

and consequently

$$\Gamma_s(-is) = \frac{1-i}{2} - \frac{1-i}{2} \sum_{j=1}^{\infty} \frac{P_{j-1}(i)}{(2s)^j} . \tag{28}$$

Then if we set $P_{j-1}(i) = A_{j-1} + i\,B_{j-1}$ and $\Gamma_s(-is) = \Pi_s(s) - i\,\Omega_s(s)$, we can deduce the expansions

$$\Pi_s(s) = \frac{1}{2} - \frac{1}{2} \sum_{j=1}^{\infty} \frac{A_{j-1} + B_{j-1}}{(2s)^j} , \tag{29}$$

$$\Omega_s(s) = \frac{1}{2} - \frac{1}{2} \sum_{j=1}^{\infty} \frac{A_{j-1} - B_{j-1}}{(2s)^j} . \tag{30}$$

When expanded to ten terms these series are explicitly:

$$\begin{aligned} 2\Pi_s(s) = 1 - \frac{1}{2s} + \frac{1}{(2s)^2} + \frac{3}{(2s)^3} - \frac{55}{(2s)^4} + \frac{599}{(2s)^5} \\ - \frac{5823}{(2s)^6} + \frac{49595}{(2s)^7} - \frac{266743}{(2s)^8} - \frac{2679473}{(2s)^9} + \cdots , \end{aligned} \tag{31}$$

$$\begin{aligned} 2\Omega_s(s) = 1 - \frac{1}{2s} + \frac{3}{(2s)^2} - \frac{13}{(2s)^3} + \frac{59}{(2s)^4} + \frac{185}{(2s)^5} \\ - \frac{1309}{(2s)^6} + \frac{45387}{(2s)^7} - \frac{832613}{(2s)^8} + \frac{12609823}{(2s)^9} + \cdots . \end{aligned} \tag{32}$$

The coefficients of the first 60 terms of these series have been calculated on the CDC 6700 system in the Computation and Mathematics Department. These numbers are listed in Tables 1 and 2. Dingle[11] has given the first four terms of the series for $\Pi_s(s)$.

Table 1. Coefficients in the Asymptotic Series $2\Pi_s(s) \sim 1 + \frac{a_1}{2s} + \frac{a_2}{(2s)^2} + \ldots + \frac{a_k}{(2s)^k} + \ldots$

k	
1	-1
2	1
3	3
4	-55
5	599
6	-5823
7	49595
8	-2 66743
9	-26 79473
10	1414 94849
11	-36721 18989
12	7 58170 24777
13	-126 64772 40185
14	1299 12747 01953
15	17923 98707 48011
16	-17 17911 68578 49399
17	716 59803 98678 89919
18	-22863 39467 40671 88479
19	5 76116 40787 21862 80803
20	-90 11585 44798 12324 81975
21	-1373 12307 50976 19279 53481
22	2 08827 66361 59497 57877 31137
23	-121 69178 34705 56823 88291 25605
24	5295 89538 42235 03570 16979 45737
25	-1 80060 61784 78714 07644 38106 73233
26	38 70985 50168 53454 95003 48440 97729
27	575 64405 63347 88761 72446 90115 85171
28	-1 32841 41270 98610 05858 45187 08904 90743
29	100 59880 30170 79671 32956 30131 03499 51975
30	-5583 41244 86055 82565 91489 36677 75596 18047
31	2 40688 17733 73565 56430 16266 63551 20113 85291
32	-66 90276 55197 81316 73940 23591 71494 76892 11319
33	-873 49040 62898 01147 70021 68810 80337 18961 51201
34	3 08000 42883 70745 45808 33044 38889 13997 13900 48641
35	-290 10482 67598 45285 57593 91632 86665 98363 80589 03357
36	19689 33992 82099 13521 69504 34830 04316 43651 44815 14185
37	-10 34072 99385 11278 31517 20921 97977 24902 35095 70716 56681
38	356 86003 97030 06463 95679 13958 18354 95970 27925 04263 91617
39	3588 99401 42698 42841 94933 08667 97198 33484 27823 51377 51675
40	-20 64813 95753 43537 60156 72538 37790 22993 00971 16709 49585 96343
41	2349 60566 39679 01453 08948 61122 67508 16108 92946 38376 22570 19407
42	-1 89473 27778 42918 87985 91212 55888 86504 68617 79745 75943 85298 38271
43	117 91722 63715 57621 15102 32208 14097 06618 23380 34471 85365 60087 25491
44	-4907 28266 58743 83887 14849 19447 01286 51575 12453 28196 37606 73405 72663
45	-29797 28133 29637 10521 39806 95000 06566 15158 20186 13645 48786 56528 43705
46	340 64847 73622 23828 31451 06589 35986 33707 83513 35598 02634 32491 29789 79713
47	-45885 73677 84107 80684 30731 45132 94459 49454 86234 99213 90637 35756 08895 33909
48	43 05638 64711 71002 77183 42821 19221 44479 49301 90002 02124 26970 21302 56795 62441
49	-3110 91560 27407 29951 86287 82994 84793 07137 69780 55510 09573 06512 50598 50285 99361
50	1 52730 79571 90567 01688 74512 39868 91212 92923 03511 39709 64279 59443 81626 96170 00961
51	1 91322 22017 39457 89638 47636 98943 98465 88405 54367 41276 93922 13890 71451 39287 29123
52	-12280 53122 55014 72109 45158 53605 55350 76773 86800 63743 91838 04909 08616 66713 36514 06775
53	19 29474 74559 70016 33925 96926 65353 85197 52172 92567 25779 87742 11522 07835 74319 54520 39799
54	-2073 66136 07247 90292 32251 73859 65224 47290 80849 25190 33914 10336 57622 79532 48551 99786 54783
55	1 71230 94592 20203 94758 19900 48941 31813 56579 44574 68805 10624 16846 17523 37980 46532 88713 05435
56	-97 48060 42776 36270 36599 43865 77461 66159 36041 92477 73953 66847 87021 65290 62340 90075 29500 40183
57	417 49016 24812 41456 47834 59374 16700 43255 46624 92017 90962 54729 97237 17768 25801 10013 43198 48047
58	8 82988 57479 96115 00047 04993 57741 02488 46439 60048 96198 75897 80145 41578 54541 94556 33066 30883 07649
59	-1600 75508 90374 34137 17615 59182 55519 41104 42840 07901 60435 37033 65376 52020 88695 73923 34392 07823 04429
60	1 94624 64634 90560 13581 39582 74924 00188 06076 33146 81917 88932 32355 46478 98704 80577 75033 64438 87627 40617

Table 2—Coefficients in the asymptotic series $2\Omega_s(s) \sim 1 + \frac{b_1}{2s} + \frac{b_2}{(2s)^2} + \ldots + \frac{b_2}{(2s)^k} + \ldots$

k	
1	-1
2	3
3	-13
4	59
5	-185
6	-1309
7	45387
8	-8 32613
9	126 09823
10	-1585 44573
11	12292 07011
12	1 50570 55675
13	-109 97350 18057
14	3710 31774 22819
15	-97331 59462 67205
16	20 35465 18552 11419
17	-262 34769 05605 13521
18	-3874 95246 92383 40157
19	4 72305 69051 68228 81747
20	-235 62407 44709 18718 94981
21	8875 23874 61660 79837 00775
22	-2 62358 66661 97428 97245 67069
23	48 57059 91475 85708 16302 43627
24	740 93368 72921 48923 85310 98267
25	-1 39099 86832 33864 03020 66299 25377
26	93 03390 26282 16638 75960 16052 16067
27	-4601 75759 56346 23548 54556 08635 32349
28	1 77240 26703 25442 25816 19923 05477 53915
29	-43 59018 76303 41339 84392 44776 66009 27337
30	-615 76092 33188 27345 02887 51595 71341 63741
31	1 74959 86777 99747 31890 62842 65491 97959 54075
32	-148 41051 46152 42851 24862 70236 81614 35137 42181
33	9147 50507 70271 89509 45876 26680 58176 20147 76559
34	-4 37028 52738 33341 66805 33455 17658 22379 10509 78877
35	135 92842 50114 88023 52209 29284 72900 62988 94235 92627
36	1589 14129 61157 47470 18682 11318 39244 57202 58536 49979
37	-7 06297 67842 72401 48588 07380 35804 11391 49380 37207 85145
38	733 41235 57018 54905 83030 13823 21644 95347 20648 81845 36291
39	-54421 81228 22082 04076 45680 88650 36250 17577 00924 54428 56693
40	31 20490 47916 37097 64302 19353 83331 57005 71059 23618 93066 54107
41	-1186 26960 90930 46670 66941 44871 16369 24714 76743 69220 24778 05857
42	-9700 67408 74282 53374 26028 16667 71292 66894 11398 64509 94755 16093
43	75 55624 61596 27088 87348 76333 62307 80466 74007 07305 45567 01151 85571
44	-9374 41995 89447 16499 99625 07533 25982 05381 70423 37314 14820 66732 63685
45	8 17290 01061 70519 73995 37103 09357 24501 06069 02025 69904 76130 88140 87543
46	-549 26176 21184 89669 72576 38017 51793 15292 20389 96705 31055 08323 51763 25661
47	24887 20967 23615 13375 03777 10684 27537 26545 41957 76776 71543 38864 98278 54715
48	93540 88016 13077 37377 82349 27323 80133 74690 63134 11720 17070 31499 12983 16699
49	-1866 51679 92401 03534 41768 84369 86584 31956 28789 73889 96469 57774 96815 10170 75761
50	2 71963 02777 46247 19320 84231 18998 35109 57980 66996 64097 73474 91556 67510 12596 82243
51	-273 58437 57829 89528 33305 63746 66076 84141 11205 23363 30723 01135 46334 30226 22748 52973
52	21168 60204 99068 86703 34338 87272 51841 86990 05167 25652 78299 86749 50279 27210 46854 98939
53	-11 21274 92550 73261 96496 67414 17165 91459 04186 99590 75554 18029 08804 29939 57148 70249 33145
54	16 05532 41939 37968 34242 70004 31365 48344 48871 45469 16522 62362 64747 37709 15150 92563 71171
55	95991 70184 71188 72077 86041 24401 03812 23277 86490 22975 16342 60260 72789 04694 32585 41755 09227
56	-162 19241 31412 91972 60607 60673 12843 34147 08145 74273 37312 82273 80902 23331 26893 77107 95707 92293
57	18665 72464 89004 13535 89252 42967 24215 93218 78802 14093 81038 13998 47917 38411 16185 57876 33804 56383
58	-16 30997 62386 74849 55533 21169 05025 88471 11338 80884 13762 10740 25401 65085 92098 01757 16985 18956 91453
59	994 50021 05358 29807 32006 24397 02453 78492 47166 61160 28679 14998 42716 89544 12465 76381 25971 14467 94371
60	-7249 05537 14774 98649 38846 60138 89944 35077 12686 30741 77085 09661 43777 34047 12143 93197 12535 34266 30725

CALCULATION OF THE CONVERGING FACTORS AND THEIR DERIVATIVES

The recurrence formula for $\Gamma_n(z)$ implies that

$$\Gamma_n(-ix) = 1 - i\left(\frac{n+1}{x}\right)\Gamma_{n+1}(-ix) \tag{33}$$

from which we infer in particular that

$$\Pi_s(s) = 1 - \left(1 + \frac{1}{s}\right)\Omega_{s+1}(s) \quad , \tag{34}$$

$$\Omega_s(s) = \left(1 + \frac{1}{s}\right)\Pi_{s+1}(s) \quad . \tag{35}$$

Clearly, if we can develop a Taylor-series expansion of $\Pi_s(s+h)$ and of $\Omega_s(s+h)$ which is convergent for $|h| \leq 1$, then we can derive numerical values of $\Pi_{s+1}(s)$ and $\Omega_{s+1}(s)$ from those of $\Pi_{s+1}(s+1)$ and $\Omega_{s+1}(s+1)$, and thence deduce values of $\Pi_s(s)$ and $\Omega_s(s)$. For large s, the values of $\Pi_s(s)$ and $\Omega_s(s)$ are obtainable from the asymptotic series (29) and (30). This iterative process can then be used to derive a table of values of these converging factors for consecutive integral values of s in descending order.

Thus, there remains the derivation of the Taylor series for $\Pi_s(s+h)$ and $\Omega_s(s+h)$, which are required for interpolation in the tables.

First, we observe that

$$\begin{aligned} \frac{d}{dz}\Gamma_s(-iz) &= -i\,\frac{d}{d(-iz)}\,\Gamma_s(-iz) \\ &= \left(\frac{s+1}{z} - i\right)\left(\Pi_s(z) - i\,\Omega_s(z)\right) + i \quad , \end{aligned} \tag{36}$$

whence

$$\frac{d}{dz}\,\Pi_s(z) = \mathrm{Re}\,\frac{d}{dz}\,\Gamma_s(-iz)$$

$$= \frac{s+1}{z}\,\Pi_s(z) - \Omega_s(z) \tag{37}$$

and

$$\frac{d}{dz}\,\Omega_s(z) = -\,\mathrm{Im}\,\frac{d}{dz}\,\Gamma_s(-iz)$$

$$= \frac{s+1}{z}\,\Omega_s(z) + \Pi_s(z) - 1 \quad . \tag{38}$$

If we set $d_j = \dfrac{d^j}{dz^j}\,\Pi_s(z)$ and $\delta_j = \dfrac{d^j}{dz^j}\,\Omega_s(z)$, evaluated at z=s, then these relations can be written

$$d_1 = (1 + \frac{1}{s})\,d_0 - \delta_0 \tag{39}$$

$$\delta_1 = (1 + \frac{1}{s})\,\delta_0 + d_0 - 1 \tag{40}$$

Next we differentiate both sides of the equation

$$z\,\frac{d}{dz}\,\Gamma_s(-iz) = [(s+1) - iz][\Pi_s(z) - i\,\Omega_s(z)] + iz \tag{41}$$

with respect to z and then set z=s. This yields the equation

$$d_1 - i\,\delta_2 + s(d_2 - i\,\delta_2) = -\,i(d_0 - i\,\delta_0) + (s+1-is)(d_1 - i\,\delta_1) + i \quad , \tag{42}$$

which implies

$$d_2 = d_1 - \delta_1 - \frac{\delta_0}{s} \tag{43}$$

and

$$\delta_2 = d_1 + \delta_1 - \frac{1 - d_0}{s} \quad . \tag{44}$$

In a similar manner we find

$$d_3 = (1 - \frac{1}{s})d_2 - \delta_2 - \frac{2}{s}\,\delta_1 \;, \tag{45}$$

$$\delta_3 = (1 - \frac{1}{s})\delta_2 + d_2 + \frac{2}{s}\,d_1 \;; \tag{46}$$

and in general

$$d_k = (1 - \frac{k-2}{s})d_{k-1} - \delta_{k-1} - \frac{k-1}{s}\,\delta_{k-2} \;, \tag{47}$$

$$\delta_k = (1 - \frac{k-2}{s})\delta_{k-1} + d_{k-1} + \frac{k-1}{s}\,d_{k-2} \;, \tag{48}$$

valid for $k \geq 3$.

Then the Taylor series for $\Pi_s(s+h)$ and $\Omega_s(s+h)$ can be written

$$\Pi_s(s+h) = d_0 + d_1 h + \frac{d_2}{2!}h^2 + \frac{d_3}{3!}h^3 + \cdots \;, \tag{49}$$

$$\Omega_s(s+h) = \delta_0 + \delta_1 h + \frac{\delta_2}{2!}h^2 + \frac{\delta_3}{3!}h^3 + \cdots \;. \tag{50}$$

Thus we have computed appended Table 3 of the converging factor $\Pi_s(s)$, designated by d_0, and its reduced derivatives $d_j/j!$ (designated as D_j) to 35 decimal places for $s = 2(1)70$, that is, for all integer values of s from 2 to 70, inclusive.

As useful by-products, there evolved tables of the converging factors $\Pi_{s+1}(s)$, $\Omega_s(s)$, and $\Omega_{s+1}(s)$, which have been rounded to 33 decimal places, and appear as Tables 4-6, respectively.(Appendixes B and C),.

For small values of z and even integer values of s, the following series given by Dingle[11] is useful:

$$\Pi_s(z) = \frac{z^2}{s(s-1)} - \frac{z^4}{s(s-1)(s-2)(s-3)} + \cdots + (-1)^{(s-2)/2}\frac{z^s}{s!}$$

$$+ \frac{(-1)^{s/2} z^{s+1}}{s!}\left\{\frac{\pi}{2}\cos z - \sum_{k=0}^{\infty}\frac{(-1)^k z^{2k+1}(\psi(2k+1)-\ln z)}{(2k+1)!}\right\}, \tag{51}$$

where $\psi(n) = -\gamma + 1 + \frac{1}{2} + \cdots + \frac{1}{n}$, $\psi(0) = -\gamma$, and γ is Euler's constant.

For odd integer values of s, Dingle gives the companion series

$$\Pi_s(z) = \frac{z^2}{s(s-1)} - \frac{z^4}{s(s-1)(s-2)(s-3)} + \cdots + (-1)^{(s-3)/2}\frac{z^{s-1}}{s!}$$

$$+ \frac{(-1)^{(s-1)/2} z^{s+1}}{s!}\left\{\frac{\pi}{2}\sin z + \sum_{k=0}^{\infty}\frac{(-1)^k z^{2k}(\psi(2k)-\ln z)}{(2k)!}\right\}. \tag{52}$$

When z=1 these series simplify appreciably; thus, we obtain

$$\Pi_1(1) = \frac{\pi}{2}\sin 1 + \psi(0) - \frac{\psi(2)}{2!} + \frac{\psi(4)}{4!} - \frac{\psi(6)}{6!} + \cdots , \tag{53}$$

$$2\Pi_2(1) = 1 - \frac{\pi}{2}\cos 1 + \psi(1) - \frac{\psi(3)}{3!} + \frac{\psi(5)}{5!} - \cdots ; \tag{54}$$

and from Equation (35) we obtain the relation, $2\Pi_2(1) = \Omega_1(1)$.

The corresponding numerical values to 45D, obtained from evaluating 20 terms of each of these series, are:

$\Pi_1(1) = 0.34337\ 79615\ 56427\ 03283\ 25330\ 03858\ 31243\ 40012\ 44019,$

$\Omega_1(1) = 0.37855\ 03757\ 64186\ 64236\ 07342\ 71784\ 66067\ 61068\ 35323.$

The values of these converging factors when computed to about 35 decimal places on the CDC 6700 system, as the final step in the descending iterative procedure beginning with $\Pi_{70}(70)$ and $\Omega_{70}(70)$, differed from the above values by less than $1 \cdot 10^{-33}$.

APPLICATIONS

In the approximation of a function with a discontinuity by a finite number of terms of its expansion in a Fourier series, a paradoxical situation occurs that is known as the Gibbs phenomenon, named after the physicist J.W. Gibbs, who first described it in 1898.

To illustrate this phenomenon consider the Fourier series $\sum_{n=1}^{\infty} \frac{\sin nt}{n}$ which has the values $(\pi - t)/2$ for $0 < t < \pi$, $(t - \pi)/2$ for $-\pi < t < 0$, and 0 for $t = 0$ or π. The approximating curves $y_n = \sum_{\nu=1}^{n} \frac{\sin \nu t}{\nu} = f_n(t)$ behave in an extraordinary manner; namely, near zero the n^{th} curve remains above the highest point, $\frac{\pi}{2}$, of the straight line by a definite ratio $K = \frac{2}{\pi} \int_0^{\pi} \frac{\sin t}{t} dt$, which is called Gibbs' constant.

To evaluate this constant to high precision, we note that

$$K = \frac{2}{\pi} \mathrm{Si}(\pi) = 1 + \frac{2}{\pi} \mathrm{si}(\pi) , \tag{55}$$

and then calculate $\mathrm{si}(\pi)$ by means of the truncated asymptotic series

$$\mathrm{si}(\pi) = \pi^{-1} - 2\pi^{-3} + 24\pi^{-5} \Pi_4(\pi) . \tag{56}$$

The converging factor $\Pi_4(\pi)$ can be calculated from the Taylor series

$$\Pi_4(\pi) = \Pi_4(4) + d_1 h + \frac{d_2}{2!} h^2 + \cdots , \tag{57}$$

where $h = \pi - 4$, and the coefficients are the reduced derivatives of $\Pi_s(s)$ evaluated at $s = 4$. If 43 terms of this series are summed, each carried to 35 decimal places, there results the value

$\Pi_4(\pi) = 0.34852\ 97445\ 99239\ 66719\ 19407\ 36171\ 10389.$

Multiplying this converging factor by $24\pi^{-5}$ and adding the product to $\pi^{-1} - 2\pi^{-3}$, we obtain

$$\mathrm{si}(\pi) = 0.28114\ 07251\ 87569\ 55112\ 97316\ 78518\ 4\ldots\ .$$

As a check on the accuracy of this approximation, the value of $\mathrm{si}(\pi)$ was recalculated by means of the series

$$\mathrm{si}(\pi) = \frac{\pi}{2} - \frac{\pi^3}{3\cdot 3!} + \frac{\pi^5}{5\cdot 5!} - \frac{\pi^7}{7\cdot 7!} + \cdots \ , \tag{58}$$

of which a total of 27 terms sufficed to give the 45-place value

$$\mathrm{si}(\pi) = 0.28114\ 07251\ 87569\ 55112\ 97316\ 78518\ 24000\ 15518\ 69618.$$

This shows that the value computed by means of the asymptotic series as modified by the introduction of an appropriate converging factor is in error by less than $2 \cdot 10^{-31}$.

The value of Gibbs' constant is then calculated to be

$$K = 1.17897\ 97444\ 72167\ 27023\ 20288\ 45824\ 90979\ 25874\ 21979\ldots$$

to 45 decimal places. This confirms the accuracy of a value calculated to 24 decimal places by D. H. Lehmer[12].

As a by-product of this calculation we infer from the relation in Equation (5) that

$$\mathrm{Si}(\pi) = 1.85193\ 70519\ 82466\ 17036\ 10533\ 70157\ 99144\ 36504\ 54318\ldots\ .$$

A companion calculation consists of evaluating $\mathrm{Ci}(\pi)$ by the truncated asymptotic series

$$\mathrm{Ci}(\pi) = \pi^{-2} - 6\pi^{-4}\,\Pi_3(\pi) \quad , \tag{59}$$

where the converging factor can be evaluated from the Taylor series

$$\Pi_3(\pi) = \Pi_3(3) + d_1 h + \frac{d_2}{2!} h^2 + \cdots \quad , \tag{60}$$

and $h = \pi - 3$. Here the coefficients $d_k/k!$ are the reduced derivatives of $\Pi_s(s)$ evaluated at $s = 3$. A total of 24 terms of this series yield the value

$$\Pi_3(\pi) = 0.44894\ 66750\ 45702\ 27893\ 45552\ 80548\ 12960\ .$$

If we multiply this converging factor by $6\pi^{-4}$ and subtract the product from π^{-2}, we obtain

$$\mathrm{Ci}(\pi) = 0.07366\ 79120\ 46425\ 48599\ 01009\ 65230\ 0\ldots\ .$$

As a check, a total of 27 terms of the series

$$\mathrm{Ci}(\pi) = \gamma + \ln \pi - \frac{\pi^2}{2 \cdot 2!} + \frac{\pi^4}{4 \cdot 4!} - \frac{\pi^6}{6 \cdot 6!} + \cdots \tag{61}$$

yielded the value

$$\mathrm{Ci}(\pi) = 0.07366\ 79120\ 46425\ 48599\ 01009\ 65230\ 14967\ 18698\ 77462\ ,$$

which reveals an error in the previous result of about $1.5 \cdot 10^{-31}$.

For $x = \pi$ we conclude that the calculation of Si(x) and Ci(x) to a specified high accuracy by power series involves less computation than that by asymptotic series in conjunction with converging factors.

However, for an argument such as 2π this situation is reversed. Thus, by means of the series

$$\mathrm{si}(2\pi) = -\frac{1}{2\pi} + \frac{2!}{(2\pi)^3} - \frac{4!}{(2\pi)^5} + \frac{6!}{(2\pi)^7} \Pi_6(2\pi) \tag{62}$$

we find the value

$$\mathrm{si}(2\pi) = -0.15264\ 47506\ 62268\ 16898\ 55415\ 29340\ 00201\ ,$$

after determining the converging factor to 35 decimal places from $\Pi_6(6)$, using 23 terms of the corresponding Taylor series. On the other hand, the calculation of this value by the power series requires the evaluation of 30 terms, each to at least 35 places. Again, by Equation (5) we deduce the value

$$\mathrm{Si}(2\pi) = 1.41815\ 15761\ 32628\ 45024\ 57801\ 62299\ 74943\ ,$$

to 35 decimal places.

Similarly, $\mathrm{Ci}(2\pi)$ can be calculated to 35 places from the series

$$\mathrm{Ci}(2\pi) = -\frac{1}{(2\pi)^2} + \frac{3!}{(2\pi)^4} - \frac{5!}{(2\pi)^6} + \frac{7!}{(2\pi)^8}\,\Pi_7(2\pi) \tag{63}$$

when the converging factor is evaluated to that precision from $\Pi_7(7)$, using 30 terms of the Taylor series. This result is

$$\mathrm{Ci}(2\pi) = -0.02256\ 06617\ 46346\ 06764\ 35387\ 78543\ 04643\ ;$$

it was checked by evaluating 32 terms of the corresponding power series.

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* Informally documented in CMD Technical Note 45-71, entitled "The Lehmer-Weinberger Multiprecision Integer Subroutine Package," October 1971.

APPENDIX A

VALUES OF $\Pi_s(s)$ AND OF ITS REDUCED DERIVATIVES

In this appendix are tabulated to 35 decimal places the values of the converging factor $\Pi_s(s)$ and its reduced derivatives D_i , defined as $\frac{1}{i!}\frac{d^i}{dx^i}\Pi_s(x)$ at $x = s$. This table (Table 3) has been photographically reproduced from computer sheets upon which the output was left-justified. Accordingly, the position of the decimal point for each tabular entry may be inferred from the right-hand indentation.

TABLE 3 - TABLE OF $\Pi_s(s)$ AND ITS REDUCED DERIVATIVES D_i TO 35 D FOR s = 2(1)70

S = 2

I	C SUB I
0	40391 60456 23264 61242 93339 56951 69424
1	18405 52805 84226 60047 88123 65346 64553
2	- 31749 17037 01893 24140 44936 58834 2018
3	14915 11905 90193 16353 06277 69133 520
4	14219 99540 24558 56275 37292 27621 389
5	- 78780 80840 88462 88072 80001 57228 82
6	29876 07055 61404 27253 02937 42981 66
7	- 99959 38707 91720 35272 27015 18805 8
8	31867 02613 90387 23845 27903 42895 2
9	- 10023 83295 37755 49637 61308 60196 7
10	31682 17075 27209 28651 17904 23048
11	- 10160 69307 07708 28635 41586 53946
12	33227 20961 99410 46378 89363 0504
13	- 11102 14132 42406 19888 67225 6361
14	37914 01406 90098 13763 83989 355
15	- 13224 82632 06427 04766 68164 058
16	47062 51380 19091 74048 85909 24
17	- 17063 04525 18092 81018 86033 99
18	62937 76720 97185 18185 33163 7
19	- 23584 81220 69028 50965 65796 6
20	89670 39740 15971 49404 30903
21	- 34549 03157 88150 04468 35557
22	13474 56830 51330 85498 91529
23	- 53144 02111 38118 38602 0368
24	21177 08327 00721 22615 4006
25	- 85192 45078 02162 61556 917
26	34573 69907 20721 20913 237
27	- 14145 50731 23165 95898 347
28	58313 00465 42916 21980 08
29	- 24207 93082 18400 02411 76
30	10115 92525 11252 32812 22
31	- 42527 55099 87133 47645 6
32	17981 80938 93847 85318 5
33	- 76440 35936 16466 27981
34	32658 57403 39927 48292
35	- 14019 34980 81931 76074
36	60450 02640 34164 6015
37	- 26175 47615 38067 3016
38	11379 47827 15075 4301
39	- 49657 83428 72745 246
40	21747 32747 25700 233
41	- 95564 82937 83697 70
42	47130 00220 36516 38
43	- 18630 23801 22703 12
44	82625 84481 29898 2
45	- 36747 29904 61998 8
46	16386 72198 87505 0
47	- 73259 66382 79973
48	32831 90245 37749
49	- 14748 25523 39174
50	66398 25319 4170
51	- 29957 43091 1254
52	13544 00851 6529
53	- 61354 95068 036
54	27847 02146 424
55	- 12662 03441 715
56	57675 90345 64
57	- 26316 22969 08
58	12027 21092 00
59	- 55054 54951 9
60	25239 73800 4
61	- 11588 23594 3
62	53280 62461
63	- 24531 29408
64	11309 70936
65	- 52208 8159
66	24131 2654
67	- 11167 1783
68	51738 987
69	- 23598 728
70	11143 969
71	- 51803 43
72	24106 33
73	- 11229 10
74	52358 5
75	- 24436 9
76	11415 9
77	- 53379
78	24981
79	- 11701
80	5485
81	- 2573
82	1208
83	- 567
84	267
85	- 125
86	59
87	- 27
88	13
89	- 6
90	2
91	- 1

S = 3

I	D SUB I
0	43050 94642 81085 40741 01666 92627 25099
1	13258 35246 98383 77782 73629 17844 89729
2	- 16820 54841 05703 41502 11800 32173 7465
3	50276 76456 83257 90903 94528 45706 73
4	41108 56382 57077 15632 26495 46383 58
5	- 15180 34612 18376 18923 52516 15167 87
6	37422 85848 94707 39411 02755 63914 1
7	- 78049 55634 60656 55096 69517 50684
8	14755 95156 69832 01356 74199 39462
9	- 25948 31138 33191 14833 78945 5772
10	42552 75261 16953 31899 77356 275
11	- 63529 60391 84789 04772 81621 73
12	78915 69777 91644 30914 60616 2
13	- 49549 52895 99196 30006 40966
14	- 15147 30631 62851 15999 13199
15	90385 68225 37728 16657 5038
16	- 32995 31426 64789 51594 3003
17	10413 19009 22300 28987 4796
18	- 30806 99488 62592 92105 419
19	88197 26836 12069 76009 58
20	- 24815 74932 76085 40357 60
21	69207 00450 40835 96627 4
22	- 19225 88789 50297 63025 6
23	53365 60484 59458 50326
24	- 14828 79647 85567 43245
25	41299 52424 64170 4790
26	- 11537 41913 48579 8071
27	32344 35094 07024 016
28	- 91018 68670 54844 48
29	25713 64219 23436 81
30	- 72931 80804 13879 7
31	20767 47208 40061 7
32	- 59366 11241 42752
33	17035 11621 43905
34	- 49063 51608 2677
35	14181 76251 7406
36	- 41134 67290 274
37	11971 24783 632
38	- 34951 95799 25
39	10236 51571 46
40	- 30069 78823 1
41	88584 27535
42	- 26168 76006
43	77511 0996
44	- 23017 3210
45	68519 400
46	- 20445 600
47	61147 01
48	- 18327 48
49	55048 8
50	- 16568 2
51	49964
52	- 15095
53	4569
54	- 1385
55	420
56	- 128
57	39
58	- 11
59	3
60	- 1

S = 4

I		D SUB I						
0		44554	95690	46484	46885	07495	58466	67808
1		10400	23226	45417	56348	53247	00395	70667
2	-	10474	60219	75494	69616	29661	48012	1640
3		22026	34557	28394	87790	96976	11909	22
4		16447	75006	97940	64443	98764	62808	29
5	-	45389	81576	99321	64452	54073	69632	7
6		82006	39392	42102	58645	82558	53512	
7	-	12051	28584	81101	99437	32228	45679	
8		15057	81204	60041	71227	24701	8963	
9	-	15448	77106	05234	57159	56232	931	
10		10170	86934	04116	85845	46102	83	
11		61137	15450	69142	60775	82912		
12	-	42852	00264	41185	60028	88071		
13		11691	71474	51475	73343	00507		
14	-	25866	53966	75997	23841	9826		
15		52307	06426	22721	48660	729		
16	-	10103	44948	63209	28709	416		
17		19044	15953	48877	79326	91		
18	-	35452	67051	80750	55571	6		
19		65659	56086	70126	29249			
20	-	12154	04229	00850	58316			
21		22553	95738	99988	9367			
22	-	42039	14348	58745	739			
23		78806	04605	13740	31			
24	-	14868	85217	74562	10			
25		28248	89795	48086	4			
26	-	54053	74112	35597				
27		10417	77103	89773				
28	-	20222	06094	7063				
29		39529	54146	799				
30	-	77801	23408	64				
31		15414	46172	84				
32	-	30736	05074	4				
33		61665	47302					
34	-	12445	30377					
35		25260	0453					
36	-	51549	645					
37		10575	010					
38	-	21802	36					
39		45165	0					
40	-	93991						
41		19646						
42	-	4123						
43		869						
44	-	183						
45		39						
46	-	8						
47		1						

S = 5

I		D SUB I
0		45523 70588 04917 28727 42872 20562 70980
1		85708 24288 59819 56121 90664 86225 4647
2	-	71677 67335 41081 76012 56897 20697 536
3		11266 21312 04918 53617 71493 29568 73
4		79164 99391 11338 93014 88195 69132 1
5	-	17398 53823 42393 97572 21273 38345 0
6		24629 03184 92338 90650 70386 48470
7	-	27312 12827 58467 03616 38113 2960
8		23748 43486 29823 97200 34694 236
9	-	12790 75741 70633 83057 90890 64
10	-	63406 42894 76011 25688 56902
11		34294 62211 12072 32489 61099
12	-	71822 04031 45119 84204 2115
13		11997 96014 11506 23468 9661
14	-	18029 87827 35033 82783 818
15		25492 16996 91292 81095 98
16	-	34670 26445 18613 70514 7
17		45926 02069 55554 85577
18	-	59693 75523 97773 1725
19		76458 01393 97558 737
20	-	96695 61294 72495 99
21		12074 98314 05821 57
22	-	14857 90859 15400 2
23		17927 78487 94177
24	-	21018 99451 7777
25		23529 93466 108
26	-	24234 89871 31
27		20773 07639 6
28	-	87482 6079
29	-	19849 9765
30		79406 300
31	-	19549 458
32		41326 75
33	-	81206 6
34		15308 6
35	-	28125
36		5081
37	-	907
38		160
39	-	28
40		4

S = 6

I	D SUB I
0	46200 07695 48806 52637 10099 53089 58386
1	72955 07296 53203 90317 34121 26900 7795
2	- 52200 64852 82343 94371 70191 92048 843
3	63915 44126 27590 73346 10340 99471 2
4	42980 12634 48898 74713 95086 84541 6
5	- 78352 43311 02051 28244 47408 13618
6	90712 00899 36966 90958 03480 1000
7	- 79346 19403 94529 07610 55064 565
8	49142 60749 22494 66151 18328 65
9	- 74743 35756 28030 15639 25337
10	- 38211 99562 31010 40342 27183
11	81783 31208 50613 95216 0797
12	- 11891 95244 29071 92726 4912
13	14702 70419 77061 09397 350
14	- 16485 67428 88888 90101 95
15	17203 42147 57889 69273 4
16	- 16862 01123 66908 62911
17	15474 88403 87988 8932
18	- 13034 99132 32266 720
19	94933 55569 78807 4
20	- 47412 99741 94835
21	- 14066 06184 7616
22	92317 83312 470
23	- 19135 96043 014
24	31673 66968 38
25	- 47596 55227 1
26	67914 05263
27	- 93976 5566
28	12758 9196
29	- 17116 763
30	22795 15
31	- 30229 7
32	40008
33	- 5292
34	700
35	- 92
36	12
37	- 1

S = 7

I	D SUB I							
0		46699	13110	61784	03082	68894	97670	49347
1		63540	02606	71043	92071	35748	90286	4991
2	-	39741	39989	38954	91638	93780	07062	162
3		39073	02778	47304	92543	63874	40918	7
4		25410	65652	20332	28247	34116	64389	9
5	-	39529	09780	07664	16107	51532	87784	
6		38569	22190	21476	10082	03183	4234	
7	-	27453	97634	35901	71206	92485	248	
8		12157	54004	92525	33533	18335	98	
9		25378	67229	32832	50111	08197		
10	-	13811	45044	27553	55609	06323		
11		20637	23264	23960	94230	6010		
12	-	23226	62662	08487	23270	157		
13		22441	00635	02342	72034	15		
14	-	19342	32523	08275	45593	0		
15		14918	58986	52492	37439			
16	-	99555	85516	44518	344			
17		50065	20322	12900	16			
18	-	41533	31209	73431				
19	-	36380	73883	30011				
20		70897	13551	6627				
21	-	99498	70853	955				
22		12271	13676	201				
23	-	14125	96653	76				
24		15592	14613	8				
25	-	16744	17816					
26		17648	7635					
27	-	18362	920					
28		18933	67					
29	-	19398	5					
30		19786						
31	-	2011						
32		204						
33	-	20						
34		2						

S = 8

I	D SUB I
0	47082 51123 20588 52328 73241 13458 10303
1	56296 33874 21036 57669 13643 26977 1388
2	- 31280 82175 84145 22022 83337 21935 208
3	25278 89825 27979 70353 45054 59039 8
4	16011 76916 62335 10832 36043 37558 1
5	- 21703 65455 94010 86956 58116 00911
6	18248 29790 59734 98964 73965 1468
7	- 10819 58432 09176 99855 40754 449
8	33760 92043 63845 40556 65045 5
9	20494 06461 92995 38846 20769
10	- 49827 59046 50806 51275 0194
11	58326 94482 18892 45457 190
12	- 53180 99931 13928 65879 54
13	41221 75508 60952 14032 2
14	- 27464 96987 09568 58599
15	14912 74792 80389 3049
16	- 49469 60708 59533 32
17	- 21238 82523 27573 7
18	65830 39144 97683
19	- 89621 13466 9143
20	98324 76132 461
21	- 96991 25369 61
22	89609 71759 2
23	- 79084 44704
24	67385 6036
25	- 55755 614
26	44909 26
27	- 35201 0
28	26753
29	- 1954
30	134
31	- 8

S = 9

I		D SUB I
0		47386 24328 69194 14993 42317 43801 00225
1		50546 47550 49676 30295 93601 21228 5967
2	-	25269 07945 19697 02726 52968 41988 079
3		17100 25259 51073 09134 17442 52435 7
4		10602 09133 49910 63786 81647 52560 8
5	-	12724 77200 07183 36981 53064 99599
6		93789 49639 60278 45866 87292 640
7	-	47168 13451 82771 31198 72247 21
8		99754 22190 96038 21641 73644
9		11320 94304 32542 25590 99819
10	-	19063 76683 57341 19479 4514
11		18388 90439 75480 29805 140
12	-	13946 37909 82728 68920 32
13		87933 22716 27114 99023
14	-	44607 52584 11392 6957
15		14325 19794 10779 695
16		37025 55219 00375 7
17	-	12454 52527 43804 3
18		15161 31754 13004
19	-	14459 82219 0262
20		12185 39920 593
21	-	94629 66424 4
22		68941 64620
23	-	47407 9236
24		30675 492
25	-	18391 37
26		97971
27	-	4056
28		41
29		17
30	-	2

S = 10

I		D SUB I
0		47632 79825 24009 77020 41272 49830 83659
1		45869 30025 41280 95038 79163 59913 2747
2	-	20842 29448 31042 88941 16810 21331 021
3		11992 52804 78884 66874 14559 98006 8
4		73046 43931 89132 58452 27162 30327
5	-	78622 69978 72942 45893 15455 0837
6		51498 51182 21843 34508 79301 404
7	-	22291 05827 56772 23247 80574 65
8		29413 46596 18132 53556 11674
9		59130 66521 24207 36118 4290
10	-	77973 86152 31361 40678 571
11		63832 36463 52048 08008 05
12	-	40983 58894 94778 82226 5
13		21144 03410 16966 26429
14	-	78047 95419 91028 350
15		44694 47215 09598 2
16		27359 80684 09620 4
17	-	35062 78171 18012
18		31334 64721 2359
19	-	23769 08849 426
20		16183 24550 94
21	-	10074 55245 0
22		57251 4193
23	-	28945 409
24		11937 38
25	-	25605
26	-	2042
27		386
28	-	42
29		3

S = 11

I		D SUB I						
0		47836	92135	27719	26957	07035	84154	28736
1		41988	93742	72430	90743	48207	73466	1937
2	-	17487	30026	38868	47719	24067	73235	571
3		86650	14498	00369	38461	52613	56861	
4		51995	23923	83013	38330	65485	77792	
5	-	50709	58504	53445	93673	67454	3870	
6		29847	72499	56901	52396	34586	574	
7	-	11253	98936	63695	52453	69102	33	
8		76058	13812	81512	51206	4450		
9		31076	96985	27549	77242	9270		
10	-	33976	78718	95856	33081	695		
11		24079	67980	93250	27576	61		
12	-	13254	48455	44634	40830	3		
13		55916	35874	38296	4207			
14	-	13443	74752	53322	322			
15	-	50044	30435	76785	3			
16		10112	80642	28562	7			
17	-	93168	48297	6451				
18		67362	03477	259				
19	-	42155	21357	06				
20		23456	11166	7				
21	-	11522	98688					
22		47316	870					
23	-	12772	50					
24	-	24351						
25		7545						
26	-	799						
27		66						
28	-	4						

S = 12

I	D SUB I
0	48008 69007 75239 68016 15505 64374 18352
1	38716 94961 03773 20174 28049 45791 3308
2	- 14883 38105 17504 11051 73505 82393 855
3	64198 49190 07864 26605 30933 67021
4	38033 25325 55455 18927 92634 60675
5	- 33898 76624 73589 95444 80148 3831
6	18096 50994 48868 95278 57196 585
7	- 60038 73102 81540 10679 03529 2
8	87067 31080 27876 07389 301
9	16730 78600 41657 41572 5482
10	- 15677 36674 75068 13754 827
11	97603 24775 53582 74735 3
12	- 46494 58212 16890 31099
13	15920 22389 75141 6505
14	- 17647 00619 69677 84
15	- 29568 03265 78207 5
16	34408 74663 10762
17	- 25558 11219 2265
18	15495 70941 186
19	- 81056 87561 0
20	36547 38091
21	- 13315 7072
22	27926 72
23	11224 0
24	- 20377
25	1819
26	- 131
27	8

S = 13

I	D SUB I
0	48155 22582 58320 87716 19357 50801 31532
1	35920 15362 86211 41060 00577 30174 8016
2	- 12821 59642 26484 34867 79728 32939 564
3	48594 11702 76379 15823 94279 44656
4	28471 60003 64422 94883 86047 02158
5	- 23359 01799 10684 35956 20937 2009
6	11398 43453 49813 03934 32901 551
7	- 33563 16023 57390 59313 20499 7
8	- 10056 70110 87183 97659 7292
9	92732 88009 65971 56681 410
10	- 76124 05284 27985 69524 55
11	42108 29755 36266 51355 2
12	- 17483 60896 01022 16317
13	47826 50863 28296 565
14	15341 27274 61941 3
15	- 13450 00000 21938 9
16	11808 30672 12914
17	- 73957 96018 793
18	38251 83106 75
19	- 16747 13921 2
20	59271 5075
21	- 13039 306
22	- 28594 7
23	61192
24	- 5167
25	342
26	- 19
27	1

S = 14

I		D SUB I						
0		48281	70369	79771	77667	62017	84125	92488
1		33501	70448	14687	97979	71517	64591	1962
2	-	11161	02655	51938	64268	96948	38283	930
3		37469	92485	42458	72574	66726	12714	
4		21742	16462	37823	06734	58090	41786	
5	-	16521	03460	09972	12145	43968	9152	
6		74183	56022	46279	85975	94418	58	
7	-	19531	69867	86452	29656	08826	2	
8	-	13225	00110	66631	94952	9684		
9		52943	66242	53750	82656	588		
10	-	38678	24032	71318	21032	46		
11		19184	42929	64448	45256	0		
12	-	69806	77732	12123	5025			
13		14831	93464	90783	968			
14		30872	82023	93413	3			
15	-	58127	78267	59666				
16		41928	51833	3360				
17	-	22665	21404	487				
18		10086	32072	91				
19	-	36642	13531					
20		94089	660					
21	-	75957						
22	-	20281	4					
23		18113						
24	-	1158						
25		63						
26	-	3						

S = 15

I		D SUB I
0		48391 97479 27790 70479 43700 52906 22885
1		31389 46648 28422 10283 23422 93148 7375
2	-	98037 95741 06042 99299 88582 73902 96
3		29363 19221 63880 05674 14322 13642
4		16893 09089 75428 79177 95726 92120
5	-	11951 97784 11382 92720 99502 6894
6		49670 56758 51940 89360 51284 25
7	-	11769 93269 34952 74010 07768 1
8	-	11784 39530 17482 93084 1511
9		31098 35959 95872 89839 381
10	-	20462 01692 13736 77427 06
11		91700 55285 03943 93162
12	-	29361 57862 81743 0255
13		46124 38845 04785 67
14		21122 79096 42375 8
15	-	25155 18596 22031
16		15520 24193 2765
17	-	73432 63575 36
18		28209 47360 0
19	-	83255 4693
20		12340 475
21		61533 7
22	-	74152
23		4886
24	-	258
25		11

S = 16

I		D SUB I
0		48488 96408 92146 53510 79695 22529 00881
1		29528 58924 14359 49099 68607 32923 5568
2	-	86801 88164 94524 25869 07183 40354 41
3		23340 52006 99017 64397 35026 16210
4		13326 40381 91828 77883 75933 13959
5	-	88196 05165 44967 43306 62820 437
6		34094 89264 22855 92750 97932 67
7	-	73131 29482 33004 88802 86733
8	-	93778 41754 80287 78683 688
9		18759 34036 06840 74821 188
10	-	11222 76054 76489 94782 78
11		45736 80771 89080 46605
12	-	12924 61253 78265 7183
13		13674 22783 56086 85
14		12187 61165 85997 6
15	-	11101 40340 63800
16		59960 57316 745
17	-	25058 80274 30
18		83025 07781
19	-	19084 9609
20		13727 5
21		31595 2
22	-	24251
23		1312
24	-	59
25		2

S = 17

I		D SUB I
0		48574 93298 64189 70348 41325 62883 93744
1		27876 61300 52230 76893 24025 77514 1872
2	-	77394 46109 12075 84744 74211 69370 70
3		18789 36733 25239 35480 51111 91433
4		10655 07332 34375 39890 77671 34630
5	-	66231 62770 27259 73914 97000 848
6		23923 10174 10304 59453 62946 05
7	-	46686 67022 86058 34209 79821
8	-	71349 16028 06758 78388 443
9		11598 43845 57268 63194 793
10	-	63578 63132 07646 34255 7
11		23693 74580 89866 09850
12	-	59209 39678 51242 631
13		33921 90512 14584 1
14		66881 15194 31797
15	-	50302 98663 9828
16		24146 28578 267
17	-	89684 37505 7
18		25499 18110
19	-	41843 036
20	-	93139 8
21		12945 4
22	-	7797
23		361
24	-	14

S = 18

I		D SUB I
0		48651 65733 28839 68722 26367 91275 18990
1		26400 15066 30670 26536 20133 60363 5197
2	-	69438 97684 64959 34255 33753 22328 60
3		15297 72957 07010 23280 92230 52719
4		86219 65233 67399 98897 09104 2466
5	-	50518 96464 56453 11301 51484 696
6		17117 17930 86211 15687 62495 58
7	-	30532 45551 10876 18466 95259
8	-	53281 48916 84318 45281 830
9		73357 74666 79675 54752 32
10	-	37084 68200 62660 23432 7
11		12699 33792 24730 51543
12	-	28094 31110 78751 849
13		29347 98645 04553
14		36274 72035 62491
15	-	23454 61508 6326
16		10112 89418 240
17	-	33519 20823 6
18		80977 5969
19	-	75896 64
20	-	59922 2
21		50164
22	-	2540
23		102
24	-	3

S = 19

I	D SUB I
0	48720 55106 65459 04487 07447 01341 71691
1	25072 58331 83210 80065 32490 14877 2194
2	- 62651 03986 14256 50194 55493 31318 87
3	12582 36490 14292 59910 32369 43136
4	70522 50096 80405 73597 15244 9565
5	- 39076 24444 46670 64496 14930 907
6	12463 70983 31861 56369 39093 95
7	- 20404 64211 87028 56973 54334
8	- 39535 21719 44808 72475 013
9	47377 49944 17296 68701 33
10	- 22209 98836 67535 85995 1
11	70187 41602 94601 5860
12	- 13749 46607 98458 244
13	- 49159 94682 17380
14	19756 00405 30342
15	- 11254 89939 8799
16	43936 96712 03
17	- 13028 88836 7
18	26306 4149
19	- 33666 3
20	- 30108 3
21	19297
22	- 848
23	30

S = 20

I		D SUB I
0		48782 75391 14022 43657 35804 59681 21257
1		23872 42526 57610 19839 03981 36959 0845
2	-	56812 68663 10104 57704 11409 25802 66
3		10444 71424 54122 74076 28708 67596
4		58245 85052 85903 84298 13271 6452
5	-	30608 48939 28780 53258 55703 344
6		92195 33884 01832 20934 01405 7
7	-	13905 10431 54936 69837 22196
8	-	29332 75121 88738 38986 910
9		31193 06370 31373 43716 77
10	-	13624 74043 26546 95653 5
11		39886 04491 87406 4572
12	-	69148 46119 85242 89
13	-	56908 81708 42077
14		10882 74450 45609
15	-	55536 84208 229
16		19751 24939 25
17	-	52466 28991
18		86201 081
19		76774 6
20	-	14086 6
21		7508
22	-	292
23		9

S = 21

I		D SUB I
0		48839 19475 71404 47552 31501 35649 35086
1		22782 14172 22652 05847 69514 71132 8907
2	-	51754 58035 68013 73826 85516 18204 26
3		87431 19053 49551 49404 81500 0563
4		48531 80209 74598 88281 89657 7486
5	-	24250 89838 43187 05563 64756 802
6		69178 36150 79134 03179 82848 9
7	-	96451 07774 40996 24325 7352
8	-	21835 89794 08012 51622 817
9		20905 00406 75194 49575 21
10	-	85432 37952 62951 23239
11		23247 97883 62431 9315
12	-	35616 52406 32347 58
13	-	46380 11420 84688
14		60840 98249 6494
15	-	28145 44726 811
16		91642 50365 6
17	-	21809 14502
18		27889 148
19		65104 0
20	-	64583
21		2979
22	-	103
23		2

S = 22

I		D SUB I
0		48890 63824 01013 70811 10240 30677 98415
1		21787 27933 39477 96928 20912 35012 8404
2	-	47343 54147 67594 12362 94037 18070 38
3		73748 99301 01811 58449 88931 5194
4		40763 62537 30972 74680 12025 4429
5	-	19414 50081 12261 45884 51718 591
6		52586 23435 69467 69131 76022 0
7	-	67989 75184 46012 76072 0995
8	-	16340 40555 90873 38306 904
9		14241 60045 14929 13342 18
10	-	54655 17108 25570 29398
11		13867 85083 46536 7377
12	-	18730 28959 01328 46
13	-	33839 38040 06395
14		34571 24673 9561
15	-	14628 84120 464
16		43786 06802 9
17	-	93259 5606
18		85622 75
19		40796 9
20	-	29619
21		1210
22	-	37

S = 23

I	D SUB I
0	48937 71949 97679 96182 63063 66974 33126
1	20875 81689 61957 83087 32694 13415 6155
2	- 43473 65774 86325 49479 29537 55138 03
3	62645 47441 76273 34832 80980 0902
4	34491 21107 40401 56787 73745 1750
5	- 15690 98179 60855 91376 75223 029
6	40450 85002 88727 94714 38292 5
7	- 48639 30600 36247 88188 5443
8	- 12304 86643 14317 23410 503
9	98503 56809 39946 97888 6
10	- 35616 36011 51517 09886
11	84501 79144 99761 707
12	- 10027 17907 41881 77
13	- 23563 92066 38956
14	19975 55793 7851
15	- 77867 77303 71
16	21497 51668 0
17	- 40889 7148
18	22582 35
19	23262 2
20	- 13719
21	503
22	- 14

S = 24

I	D SUB I
0	48980 97047 00699 63996 57190 50851 42793
1	20037 67378 83556 88537 45840 21972 4033
2	- 40059 84522 83795 44533 11403 70800 55
3	53558 02078 30687 37021 32988 5773
4	29381 50003 88282 57233 07910 5337
5	- 12792 64774 01219 88768 15905 845
6	31456 52369 60087 85243 16699 9
7	- 35270 87980 18535 04876 9956
8	- 93291 32755 99694 90863 92
9	69095 64905 31960 06533 4
10	- 23607 67426 43340 87349
11	52508 09341 09859 196
12	- 54487 00437 84766 8
13	- 16070 22389 72097
14	11735 80391 8724
15	- 42387 36058 61
16	10824 50647 8
17	- 18322 8948
18	31417 0
19	12793 0
20	- 6448
21	214
22	- 5

S = 25

I		D SUB I
0		49020 84001 67093 74833 06617 04568 88525
1		19264 33312 40610 46505 02201 01239 0429
2	-	37033 11683 63234 86260 75750 41468 11
3		46062 33491 70968 81197 71359 7532
4		25184 91909 10949 44475 13206 0606
5	-	10513 76877 46532 97638 74178 444
6		24708 46786 03171 35903 43905 4
7	-	25898 19484 58388 39738 7375
8	-	71227 93459 10234 52065 30
9		49104 69413 55010 76682 0
10	-	15895 98111 05806 40144
11		33223 38846 84439 164
12	-	29963 68087 68841 5
13	-	10869 23986 38096
14		70079 41995 356
15	-	23564 52757 43
16		55798 52263
17	-	83638 396
18	-	19762 6
19		69489
20	-	3082
21		93
22	-	2

S = 26

I	D SUB I
0	49057 70953 76837 24045 56530 93919 28473
1	18548 54952 50145 41388 47605 04088 5929
2	- 34337 05965 31443 95197 84435 78681 42
3	39834 85979 78554 44682 69763 3987
4	21712 27632 17165 92820 92798 1719
5	- 87052 13196 74549 15774 76426 00
6	19588 46789 61603 61287 65367 8
7	- 19237 05632 76854 29116 5972
8	- 54766 68167 76209 75558 20
9	35324 71907 84476 13089 9
10	- 10860 66732 66206 66865
11	21376 84454 64190 706
12	- 16622 52046 29926 4
13	- 73404 07876 7505
14	42510 37764 080
15	- 13362 03745 92
16	29398 61577
17	- 38756 065
18	- 26612 9
19	37712
20	- 1500
21	41

S = 27

I		D SUB I
0		49091 90517 61210 76504 23300 48626 79272
1		17884 12012 88557 99914 06348 22229 7667
2	-	31925 18032 89313 75868 76679 09977 26
3		34626 31806 39436 44185 33716 3674
4		18818 61344 80719 54428 22464 3849
5	-	72575 13795 52498 34788 12332 19
6		15663 14318 65844 69747 49159 6
7	-	14443 08585 35872 07248 3532
8	-	42403 44093 01332 39164 02
9		25701 87163 03607 40121 4
10	-	75217 12243 62147 1020
11		13970 46459 45063 765
12	-	92686 84792 18714
13	-	49690 01150 3836
14		26179 05764 895
15	-	77190 38177 6
16		15807 99370
17	-	18160 067
18	-	21824 1
19		20574
20	-	743
21		19

S = 28

I		D SUE I
0		49123 70747 35576 15773 27964 69603 77968
1		17265 70340 98582 41106 00760 39556 0662
2	-	29758 88233 62998 29967 41147 32618 90
3		30242 82754 29356 71709 99123 1873
4		16391 76178 49413 67960 02975 5454
5	-	60893 82172 40912 85634 89520 21
6		12624 57795 93163 56399 94178 6
7	-	10952 37540 78649 97192 5940
8	-	33054 83324 29865 84751 94
9		18899 96215 34293 22217 1
10	-	52755 60012 99419 0030
11		92636 87113 34749 38
12	-	51717 77090 12968
13	-	33794 14631 6818
14		16356 14610 386
15	-	45378 71261 8
16		86633 0584
17	-	85655 23
18	-	15543 2
19		11321
20	-	375
21		8

S = 29

I		D SUB I
0		49153 35906 82622 69545 38436 74775 95667
1		16688 67454 77942 75853 96402 24763 3520
2	-	27805 90878 49437 37387 61884 18190 24
3		26532 25537 79300 79768 65967 4851
4		14344 12646 09793 65040 74094 5539
5	-	51398 03174 59596 41437 91700 36
6		10251 25842 59720 78428 63098 0
7	-	83828 44902 66099 30445 684
8	-	25937 62686 08427 85941 28
9		14037 06378 21906 41882 7
10	-	37441 35810 89187 3202
11		62265 37196 94196 25
12	-	28711 22376 12574
13	-	23122 29180 0266
14		10360 65577 661
15	-	27120 45897 1
16		48329 0652
17	-	40431 98
18	-	10425 5
19		6295
20	-	192
21		4

S = 30

I		D SUB I
0		49181 07088 53420 89338 89345 05311 50074
1		16149 00901 67674 00628 15653 55212 7172
2	-	26039 13219 36091 73527 38488 77663 33
3		23374 24334 20925 95400 25403 3235
4		12606 71731 36543 87372 95497 6029
5	-	43625 19063 38593 96397 43195 62
6		83819 73345 53115 57616 37667
7	-	64721 10298 29812 05337 674
8	-	20482 77782 40761 15293 60
9		10523 14819 11431 19543 7
10	-	26868 24952 54884 0165
11		42386 03016 48580 41
12	-	15728 25658 40129
13	-	15928 87125 5066
14		66495 08848 14
15	-	16462 22612 3
16		27412 9696
17	-	18946 92
18	-	67951
19		3541
20	-	100
21		2

S = 31

I		C SUB I
0		49207 02715 23966 78460 78455 01975 89199
1		15643 18816 77487 97606 29483 24020 4179
2	-	24435 60556 85426 69257 82674 29616 65
3		20672 83811 78880 15069 56636 9418
4		11124 76364 66162 91261 36317 9987
5	-	37221 32139 76215 16570 64104 32
6		68981 25649 30212 55215 07713
7	-	50377 13169 18409 64723 089
8	-	16274 50737 53380 36228 70
9		79584 25438 80797 29133
10	-	19482 08444 35835 9588
11		29199 20644 42526 85
12	-	83927 35758 0950
13	-	11053 40752 7400
14		43212 99715 97
15	-	10140 24720 4
16		15793 2418
17	-	87066 6
18	-	43691
19		2016
20	-	53
21		1

S = 32

I	D SUB I
0	49231 38949 26733 08677 70146 52381 51439
1	15168 12210 15711 43861 50350 24523 4412
2	- 22975 81221 92845 89256 74056 79384 32
3	18350 99152 35422 91466 20253 8234
4	98544 59057 39146 06992 13992 374
5	- 31913 18358 95307 27900 14743 87
6	57115 84308 91941 69745 64505
7	- 39512 78839 54157 51922 282
8	- 13007 17047 97896 89086 28
9	60687 49392 70853 18136
10	- 14264 93063 65002 5144
11	20341 40510 31500 28
12	- 42624 46445 7908
13	- 77278 05819 211
14	28418 43447 35
15	- 63332 60403
16	92328 072
17	- 38404 5
18	- 27941
19	1162
20	- 28

S = 33

I	D SUB I
0	49254 30028 77475 04929 96365 49918 96206
1	14721 08624 52238 53438 31337 48402 5023
2	- 21643 06811 21813 69986 06446 84937 09
3	16346 41766 27841 67740 24618 8693
4	87605 21634 18285 00213 15873 356
5	- 27488 13232 56029 99284 14037 90
6	47562 37122 36742 55598 89017
7	- 31214 87924 74965 68302 468
8	- 10454 66792 12518 18623 40
9	46640 02146 30456 30257
10	- 10541 33005 41959 6191
11	14320 85828 61581 35
12	- 19599 18401 7255
13	- 54436 57372 066
14	18901 59360 11
15	- 40077 58688
16	54721 078
17	- 15554 3
18	- 17858
19	677
20	- 15

S = 34

I		D SUB I
0		49275 88545 73978 09362 47214 88152 28354
1		14299 66887 08450 87467 94994 21642 6541
2	-	20423 04230 62182 53616 18741 45746 25
3		14608 44474 31139 92154 96595 0798
4		78143 48647 01708 39338 52771 737
5	-	23779 40921 19331 87633 13228 07
6		39820 48412 80414 88812 52417
7	-	24827 07214 17577 71529 194
8	-	84486 79419 26381 84208 5
9		36109 30329 92887 17764
10	-	78575 42722 40787 801
11		10183 00668 02152 47
12	-	70137 48023 245
13	-	38635 08997 983
14		12707 91411 40
15	-	25678 62613
16		32852 841
17	-	51023
18	-	11441
19		400
20	-	8

S = 35

I	D SUB I
0	49296 25677 11893 21354 25690 27815 65169
1	13901 72741 49796 87810 72585 28002 1448
2	- 19303 36953 31483 16135 50428 84930 96
3	13095 60297 52836 96962 09126 8158
4	69926 08962 55598 87220 62311 728
5	- 20655 29610 84087 85981 45931 22
6	33508 33119 63164 74342 94522
7	- 19873 04981 87769 23284 459
8	- 68631 03762 69089 33275 5
9	28151 94523 35621 64750
10	- 59052 22851 43684 783
11	73091 01749 24590 5
12	- 37804 32187 91
13	- 27623 75515 096
14	86318 95187 7
15	- 16647 94006
16	19964 224
17	- 5657
18	- 7360
19	239
20	- 4

S = 36

I		D SUB I
0		49315 51378 12711 32670 85595 20989 86310
1		13525 35192 22763 95924 48243 61415 0385
2	-	18273 33522 38304 79309 66882 07424 31
3		11773 76207 61823 56361 11936 1755
4		62761 60734 66180 57146 39652 437
5	-	18011 04444 22300 09676 23370 89
6		28332 20632 61775 42215 27327
7	-	16003 96210 45470 65545 336
8	-	56028 78032 27840 59036 3
9		22093 67797 03197 37169
10	-	44725 29396 48613 534
11		52931 50338 11400 9
12		28920 47523 875
13	-	19894 23747 601
14		59208 70272 4
15	-	10914 66042
16		12270 972
17		11955
18	-	4761
19		144
20	-	2

S = 37

I		D SUB I
0		49333 74544 68640 54336 30974 48250 68827
1		13168 83429 30490 69100 09148 67922 4751
2	-	17323 61789 77431 27053 81899 57236 03
3		10614 68260 31093 36617 78396 8075
4		56492 12913 38387 59208 47932 469
5	-	15762 81830 83747 24085 87460 68
6		24064 52314 83217 36705 64824
7	-	12962 09981 18329 18715 063
8	-	45958 98353 27451 07275 0
9		17448 22644 24004 56903
10	-	34124 12386 94639 562
11		38656 72615 33109 8
12		42866 41606 639
13	-	14429 05018 706
14		40993 62679 4
15	-	72323 8816
16		76235 58
17		16980
18	-	3098
19		88
20	-	1

S = 38

I	D SUB I
0	49351 03150 63191 75028 28147 22519 81083
1	12830 64228 68022 62674 79077 96634 9194
2	- 16446 07727 65273 98943 75192 26773 40
3	95948 81368 69608 81480 69474 535
4	50986 71012 98683 24456 77286 650
5	- 13843 11211 05138 30219 80775 30
6	20527 69221 87816 00219 95114
7	- 10555 52283 27739 25748 496
8	- 37871 47703 28689 34882 5
9	13861 88781 95019 13347
10	- 26217 94147 57893 511
11	28458 38817 53211 9
12	46639 07540 912
13	- 10537 28635 244
14	28636 12584 7
15	- 48411 9357
16	47842 20
17	16665
18	- 2030
19	54

S = 39

I		D SUB I						
0		49367	44364	13464	39962	11770	88932	41816
1		12509	39744	47207	53302	70704	93643	7727
2	-	15633	57907	40373	73375	89691	31662	30
3		86947	37022	81511	85426	22995	743	
4		46136	23206	92464	16100	55883	243	
5	-	12197	25727	21959	11171	98587	06	
6		17582	21693	74283	77719	93608		
7	-	86401	65925	63150	38477	11		
8	-	31344	02483	33889	04398	5		
9		11075	29364	60879	81044			
10	-	20277	47470	33589	008			
11		21110	45244	05935	5			
12		45153	38296	380				
13	-	77465	99457	88				
14		20174	65581	0				
15	-	32720	1348					
16		30309	35					
17		14396						
18	-	1339						
19		34						

S = 40

I	D SUB I
0	49383 04646 92299 41629 21854 62791 97367
1	12203 85625 84210 40689 65145 01811 3279
2	- 14879 84938 11764 83143 22971 25256 55
3	78977 80589 37438 52891 95709 686
4	41849 34074 78978 29418 02900 806
5	- 10780 73992 18547 26236 70599 84
6	15117 83266 12386 07083 76003
7	- 71071 01564 82342 75665 26
8	- 26050 74675 31057 57474 7
9	88967 77786 63156 0599
10	- 15782 20424 17235 449
11	15773 45909 84875 6
12	41191 99652 846
13	- 57318 34742 31
14	14329 43991 7
15	- 22319 0792
16	19373 33
17	11691
18	- 889
19	21

S = 41

I	D SUB I
0	49397 89839 19627 91929 83778 95087 59562
1	11912 89404 11322 34620 25858 86782 4946
2	- 14179 35306 61254 77729 71864 25507 06
3	71901 29399 55465 24564 12905 261
4	38049 21507 39706 05100 38579 628
5	- 95571 27163 19963 93686 82016 6
6	13046 86009 09359 55066 44829
7	- 58733 94076 66828 65272 95
8	- 21738 75475 12460 29192 0
9	71836 60312 28852 3004
10	- 12357 48651 10575 015
11	11867 24717 17668 3
12	36298 28696 090
13	- 42676 15723 58
14	10257 23991 0
15	- 15358 7343
16	12486 93
17	9172
18	- 595
19	13

S = 42

I		D SUB I
0		49412 05232 58039 58297 88036 16906 96863
1		11635 49105 92075 14942 34720 24711 0135
2	-	13527 19176 68699 37606 23378 92930 61
3		65600 32912 92411 28602 34178 612
4		34670 98271 81833 97489 63839 809
5	-	84964 53418 45784 93799 46381 6
6		11299 18305 30527 34690 68356
7	-	48754 77800 94297 88972 57
8	-	18210 76608 43644 87816 1
9		58289 81477 74571 8152
10	-	97315 15327 36799 43
11		89872 83500 92754
12		31298 26804 924
13	-	31966 55931 97
14		73971 79021
15	-	10658 1950
16		81115 1
17		7050
18	-	401
19		8

S = 43

I	D SUB I
0	49425 55633 04492 13999 42738 65654 14962
1	11370 72056 26715 72363 01413 89684 5987
2	- 12919 01794 89230 45333 39993 86989 37
3	59975 06372 59164 95249 16230 046
4	31659 64167 22541 40291 95354 210
5	- 75739 57008 52590 21490 63684 2
6	98184 27251 53033 29287 1793
7	- 40643 24761 77096 74261 98
8	- 15312 09015 31375 36118 2
9	47520 21755 60337 4809
10	- 77055 90812 62702 63
11	68490 98810 05070
12	26606 38665 730
13	- 24084 46797 64
14	53727 99590
15	- 74559 476
16	53078 6
17	5351
18	- 272
19	5

S = 44

I	D SUB I
0	49438 45415 35673 77365 82549 78154 77461
1	11117 73841 80632 01833 00742 03856 7361
2	- 12350 96219 98672 00833 16264 87799 47
3	54940 33811 06669 67664 27088 178
4	28968 37998 93769 08896 48012 450
5	- 67690 85131 00017 34758 33920 0
6	85590 33530 78976 38333 6713
7	- 34018 91676 37384 81842 75
8	- 12920 83424 48024 23847 3
9	38914 74487 25953 9219
10	- 61334 00322 61662 12
11	52510 31425 19071
12	22404 69057 405
13	- 18248 31327 95
14	39292 21567
15	- 52561 135
16	34969 9
17	4031
18	- 185
19	3

S = 45

I	D SUB I
0	49450 78570 36907 99318 19648 88641 14051
1	10875 77409 86887 10961 18768 46619 8629
2	- 11819 57148 05556 56137 53022 81440 01
3	50423 26115 23522 97912 99191 764
4	26557 21069 34976 92116 68612 876
5	- 60647 05823 96052 20665 10469 2
6	74840 02103 20673 84099 3683
7	- 28584 88846 72519 89385 88
8	- 10940 49166 49417 92154 7
9	32004 92227 28914 9063
10	- 49064 58529 45803 97
11	40490 15624 43161
12	18747 50141 321
13	- 13901 74994 00
14	28924 34796
15	- 37327 608
16	23185 2
17	3024
18	- 127
19	2

S = 46

I		D SUB I
0		49462 58746 22200 39962 45665 46520 16470
1		10644 12282 92776 08814 29672 70105 8162
2	-	11321 75648 54387 20358 83491 94802 72
3		46361 23034 17955 10924 06931 840
4		24391 85749 72244 77262 17859 503
5	-	54464 80519 59398 89140 81570 4
6		65631 43451 62888 53886 2101
7	-	24108 18389 50500 50543 68
8	-	92943 02919 62116 09430
9		26430 66926 58102 7935
10	-	39437 93746 06156 45
11		31393 70170 76867
12		15622 14851 032
13	-	10646 25080 15
14		21426 85678
15	-	26697 363
16		15461 8
17		2263
18	-	88
19		1

S = 47

I		D SUB I
0		49473 89284 34949 04933 10736 35143 77827
1		10422 13871 60175 90553 42087 89872 7192
2	-	10854 74660 77692 49026 52430 73834 71
3		42700 30434 25907 55331 65971 874
4		22442 84113 94070 04485 57060 720
5	-	49023 61909 84060 53998 06996 2
6		57717 14304 55176 56135 6448
7	-	20405 01958 68361 34151 21
8	-	79209 43622 07010 84240
9		21913 66742 41792 2489
10	-	31845 86168 71180 02
11		24469 49331 52061
12		12983 77278 830
13	-	81945 80754 7
14		15969 23825
15	-	19224 556
16		10366 5
17		1692
18	-	61
19		1

S = 48

I	D SUB I
0	49484 73251 04108 63314 25400 28085 51368
1	10209 22872 06596 73529 20881 95116 9855
2	- 10416 05127 92825 01697 85170 43294 71
3	39393 85962 49934 78434 64889 454
4	20684 72698 67150 31812 15363 324
5	- 44221 90134 62640 01802 17782 3
6	50893 48063 38352 88362 3188
7	- 17329 68912 54809 47943 93
8	- 67712 10412 69644 02498
9	18237 63850 52222 4707
10	- 25828 74651 65789 48
11	19169 19824 82605
12	10774 85872 578
13	- 63384 77411 5
14	11971 25594
15	- 13934 074
16	69841
17	1266
18	- 43

S = 49

I	D SUB I
0	49495 13465 18539 13594 73493 15978 46159
1	10004 84736 00870 26836 51230 33430 7069
2	- 10003 42667 34866 84223 57719 75909 39
3	36401 47711 62252 12149 76122 733
4	19095 50287 44448 63769 49147 362
5	- 39973 66522 60827 57807 11543 0
6	44992 11709 36838 06078 2956
7	- 14766 11653 65067 04752 23
8	- 58054 61501 49899 87550
9	15233 64813 32785 9573
10	- 21037 24080 74980 17
11	15090 10038 37998
12	89358 18959 70
13	- 49260 59205 7
14	90246 3137
15	- 10163 060
16	47258
17	948
18	- 30

S = 50

I		D SUB I
0		49505 12522 72354 23803 22919 86150 73791
1		98084 92031 41983 72152 86925 78264 128
2	-	96148 46938 55729 08880 99823 14262 2
3		33688 01591 12780 50789 59122 169
4		17656 06260 47908 91552 12240 346
5	-	36205 88810 41646 23634 89820 2
6		39873 37887 74582 70125 8507
7	-	12621 40259 45488 24719 32
8	-	49916 30668 07038 16711
9		12769 08855 01918 1305
10	-	17204 31643 64487 75
11		11934 58453 88439
12		74103 87893 66
13	-	38459 35741 7
14		68401 0970
15	-	74574 84
16		32099
17		712
18	-	21

S = 51

I		D SUB I
0		49514 72818 25898 13068 09560 07561 79637
1		96196 98878 16085 34708 28531 63049 866
2	-	92484 79268 95654 07267 74331 80273 3
3		31222 83975 32289 54671 86293 290
4		16349 77553 09057 77814 91414 070
5	-	32856 35333 50116 67609 62969 0
6		35420 93217 33794 87774 7291
7	-	10820 86670 50478 51175 92
8	-	43036 79178 00286 31994
9		10739 37171 22575 7170
10	-	14124 77242 87447 09
11		94813 89334 7438
12		61480 52572 55
13	-	30159 64620 2
14		52114 0594
15	-	55040 73
16		21874
17		535
18	-	15

S = 52

I		D SUB I
0		49523 96564 20193 11734 91231 66810 53519
1		94380 39125 68731 47936 85836 49898 094
2	-	89026 62241 24282 58584 17449 79763 0
3		28979 16878 40839 07796 59745 788
4		15162 12657 12598 68874 75059 191
5	-	29871 88406 83200 44194 15577 1
6		31537 53081 88141 92824 5711
7	-	93042 17669 91199 95596 2
8	-	37203 83699 58339 03970
9		90616 31195 96007 301
10	-	11640 10521 68073 95
11		75650 71156 9127
12		51048 37328 66
13	-	23752 49128 2
14		39904 6823
15	-	40851 48
16		14945
17		403
18	-	11

S = 53

I		D SUB I
0		49532 85807 77053 21910 52304 52043 43741
1		92631 15825 05414 83106 42514 43262 668
2	-	85758 86934 50708 93380 03377 47720 9
3		26933 53441 71797 30260 66468 675
4		14080 41407 52659 79771 60691 719
5	-	27206 89207 06667 36794 58202 8
6		28141 60066 12995 28396 5362
7	-	80225 81312 92315 28671 7
8	-	32243 83213 20477 46605
9		76699 21642 23718 567
10	-	96272 73576 23056 6
11		60612 37727 9703
12		42432 14043 05
13	-	18784 09874 2
14		30703 8610
15	-	30484 28
16		10232
17		305
18	-	8

S = 54

I	D SUB I
0	49541 42446 12342 44439 43329 24995 41189
1	90945 60952 98166 20246 66522 20727 762
2	- 82667 80433 71440 60682 62626 22188 9
3	25065 31941 61525 16160 36042 171
4	13093 49538 95350 52966 47601 969
5	- 24822 18084 36301 65921 69563 3
6	25164 48410 29568 27290 1099
7	- 69361 81960 64248 84049 5
8	- 28014 26449 63916 05055
9	65115 40659 09649 325
10	- 79903 07889 78660 6
11	48758 63405 8935
12	35315 84710 69
13	- 14914 61237 5
14	23735 0020
15	- 22866 83
16	7014
17	231
18	- 5

S = 55

I	D SUB I
0	49549 68239 85903 10595 08390 42362 44933
1	89320 32823 37236 05500 55595 70933 244
2	- 79740 91378 27882 16202 29037 98115 0
3	23356 36863 13923 56505 78175 960
4	12191 57189 98347 51204 11260 432
5	- 22683 95486 69705 33013 69457 1
6	22548 20809 22080 25758 9832
7	- 60125 25595 63648 22195 1
8	- 24397 75265 94709 65249
9	55441 98093 49360 295
10	- 66540 09108 62064 9
11	39375 11049 0744
12	29435 89465 85
13	- 11888 26854 0
14	18430 7638
15	- 17239 29
16	4809
17	176
18	- 4

S = 56

I		D SUB I
0		49557 64825 08354 76174 83595 82536 77354
1		87752 13771 98904 16915 42453 96045 908
2	-	76966 77271 09905 96958 76986 89930 4
3		21790 65853 23560 72080 61529 802
4		11366 00685 76759 06320 38824 032
5	-	20762 99658 12587 41361 02253 3
6		20243 66897 41747 81048 1294
7	-	52249 69334 12433 24683 1
8	-	21297 29795 35501 18504
9		47338 23397 06618 124
10	-	55591 91851 73980 4
11		31916 36697 1806
12		24573 78732 93
13	-	95116 80423
14		14374 3834
15	-	13059 96
16		3296
17		134
18	-	3

S = 57

I	D SUB I
0	49565 33724 22158 94662 21015 33591 55724
1	86238 08081 23487 07676 03165 01001 659
2	- 74334 93307 13092 33709 94206 57200 0
3	20354 01582 88397 53117 44351 115
4	10609 17052 74058 46634 34335 423
5	- 19033 98039 24381 17206 42994 1
6	18209 15072 24878 58770 1907
7	- 45515 90760 89303 18043 3
8	- 18632 49081 82569 68224
9	40528 65149 36938 588
10	- 46590 74858 45530 8
11	25963 95856 5351
12	20549 21132 37
13	- 76379 55786
14	11258 0769
15	- 99403 7
16	2255
17	102
18	- 2

S = 58

I		D SUB I
0		49572 76355 71970 23810 77461 77165 80088
1		84775 40116 47220 79913 12246 78228 758
2	-	71835 82517 04417 95977 79283 21511 6
3		19033 87720 88854 42893 20425 770
4		99143 08175 18494 14670 27997 24
5	-	17474 89901 66574 05494 75310 0
6		16409 11088 04607 05095 7674
7	-	39742 89976 04237 44327 7
8	-	16336 46883 28869 28735
9		34789 61085 55989 494
10	-	39165 28537 16686 3
11		21195 38715 6310
12		17213 80167 04
13	-	61550 22839
14		88533 689
15	-	76003 6
16		1537
17		79
18	-	1

S = 59

I		D SUB I
0		49579 94042 77278 10361 54153 43247 20610
1		83361 52649 13646 68925 68840 17692 007
2	-	69460 67051 94990 40260 49664 97708 4
3		17819 08362 27656 83616 16696 661
4		92754 27215 91743 26265 68739 89
5	-	16066 58225 54556 58069 45608 7
6		14813 18232 24276 78880 3461
7	-	34780 73877 53673 72142 4
8	-	14353 46878 53964 26019
9		29938 96439 46484 962
10	-	33019 66093 79518 9
11		17360 99496 5044
12		14445 71061 55
13	-	49770 07349
14		69898 100
15	-	58367 7
16		1043
17		60
18	-	1

S = 60

I		D SUB I
0		49586 88021 18627 25795 93574 40548 26755
1		81994 05345 15549 94396 92574 53815 790
2	-	67201 40460 66892 16309 30285 64659 9
3		16699 70367 95554 56252 54222 922
4		86872 00477 49008 37212 96480 29
5	-	14792 29207 95943 96639 84621 7
6		13395 34966 43296 04527 4897
7	-	30504 83371 27723 26383 9
8	-	12636 84995 13847 84592
9		25827 83790 44081 084
10	-	27917 20326 04189 6
11		14266 68117 3494
12		12144 97498 07
13	-	40378 33244
14		55396 254
15	-	45015 0
16		702
17		47

S = 61

I	D SUB I
0	49593 59446 47238 33101 33833 53720 53011
1	80670 73399 96541 13778 07152 62429 586
2	- 65050 60832 41573 47130 01026 28741 2
3	15666 88164 84858 01177 63833 131
4	81448 83064 31085 40403 72528 55
5	- 13637 38091 94199 41905 84068 1
6	12133 26752 47579 97820 5663
7	- 26811 33249 32113 91787 3
8	- 11147 49251 31795 25803
9	22334 14851 55627 127
10	- 23667 88683 83776 5
11	11760 91095 6421
12	10229 62212 68
13	- 32864 51443
14	44065 822
15	- 34860 2
16	467
17	36

S = 62

I		D SUB I
0		49600 09400 26595 45093 92446 28423 97040
1		79389 46303 79575 74066 89644 80419 894
2	-	63001 44695 80274 11086 05955 47331 8
3		14712 70631 35062 79401 54046 602
4		76442 40728 27246 06678 05595 42
5	-	12589 00247 59639 99965 82356 2
6		11007 69440 52334 09789 6081
7	-	23613 41226 17780 81399 4
8	-	98524 95691 50528 8987
9		19357 46168 58798 055
10	-	20118 58876 61472 8
11		97249 04716 830
12		86324 29796 4
13	-	26832 66646
14		35178 749
15	-	27104 1
16		306
17		28

S = 63

I		D SUB I
0		49606 38896 13489 76077 25049 28941 86984
1		78148 26722 94318 70541 78134 44536 071
2	-	61047 61580 30240 50743 27175 62447 8
3		13830 09754 93698 07043 87561 313
4		71814 88003 90738 22582 40535 81
5	-	11635 86630 78778 97172 62031 0
6		10002 02113 17872 40814 3028
7	-	20838 27801 92242 23566 4
8	-	87241 16150 30996 6826
9		16814 89536 69853 017
10	-	17145 49165 56050 4
11		80651 95123 940
12		72982 49232 5
13	-	21974 49619
14		28181 802
15	-	21155 3
16		196
17		22

S = 64

I	D SUB I
0	49612 48884 85080 41646 72305 04740 31839
1	76945 29484 51464 46575 21807 08151 909
2	- 59183 29159 21055 83937 20432 54481 3
3	13012 70799 73353 05472 83773 093
4	67532 34650 69151 45176 96867 96
5	- 10768 02901 39838 75858 80318 6
6	91018 76891 24416 75609 054
7	- 18424 72600 48495 63372 4
8	- 77388 99666 15360 7939
9	14637 84629 43984 891
10	- 14648 13657 25749 9
11	67079 55495 778
12	61818 04753 4
13	- 18049 06934
14	22652 798
15	- 16574 2
16	121
17	17

S = 65

I	D SUB I
0	49618 40259 17734 38239 92045 13128 68322
1	75778 80653 64234 02175 66211 20041 822
2	- 57403 08904 11746 20024 06306 79450 2
3	12254 83763 91422 92720 27903 113
4	63564 39181 13549 99064 10653 90
5	- 99767 16097 08703 96394 59533
6	82947 99143 77082 89309 329
7	- 16321 15902 71294 76965 9
8	- 68769 68754 76497 5227
9	12769 36418 44796 304
10	- 12544 74948 49264 0
11	55946 59439 001
12	52458 91301 1
13	- 14867 41019
14	18268 231
15	- 13032 5
16	70
17	13

S = 66

I		C SUE I
0		49624 13858 22714 24361 98150 36162 55147
1		74647 16693 48896 21806 55466 93197 679
2	-	55702 02190 23137 75720 23103 01028 3
3		11551 35941 59240 98615 74185 564
4		59883 68447 99394 91343 35981 97
5	-	92541 69622 29325 09746 86751
6		75699 56281 68163 52057 037
7	-	14483 96473 87012 60734 1
8	-	61214 36545 89251 4492
9		11162 03843 35896 354
10	-	10768 55464 98251 7
11		46787 45473 664
12		44599 01465 1
13	-	12280 77390
14		14779 217
15	-	10283 8
16		37
17		10

S = 67

I		D SUB I
0		49629 70471 43183 29836 25948 97318 18918
1		73548 83699 50320 78674 00036 25247 377
2	-	54075 46799 87338 13681 73556 98099 5
3		10897 65432 75599 72886 01634 476
4		56465 62425 23728 97519 43511 65
5	-	85935 17620 02524 06505 85255
6		69179 13974 65174 57101 796
7	-	12876 18637 09184 05778 9
8	-	54579 23170 25780 6584
9		97762 91670 08992 74
10	-	92648 55884 23197
11		39230 43624 763
12		37986 23558 1
13	-	10171 68032
14		11993 531
15	-	81428
16		15
17		8

S = 68

I	D SUB I
0	49635 10842 16475 94918 65438 31833 18002
1	72482 36700 37804 17382 57903 27935 119
2	- 52519 13778 28806 16108 69612 64546 2
3	10289 55468 83723 13760 39379 134
4	53288 03451 48096 62839 87307 32
5	- 79886 61867 77820 53324 74087
6	63304 37802 96144 93127 282
7	- 11466 42990 81551 26840 8
8	- 48741 55633 33722 5678
9	85789 97415 01786 91
10	- 79887 17578 37994
11	32977 82155 853
12	32412 61682 5
13	- 84470 3292
14	97621 51
15	- 64690
16	0
17	6

S = 69

I		D SUB I
0		49640 35671 05127 56868 19801 53226 69037
1		71446 39019 02853 59145 01890 36155 664
2	-	51029 04601 77183 27059 50537 24308 7
3		97232 94415 73007 69407 01689 19
4		50330 89316 27553 62370 98236 04
5	-	74341 81245 14752 67315 95717
6		58003 26118 40254 16821 919
7	-	10227 96302 34221 42705 7
8	-	43596 35878 89017 8100
9		75423 56809 07330 13
10	-	69031 15874 41670
11		27790 37809 272
12		27706 37038 9
13	-	70328 1827
14		79691 21
15	-	51560
16	-	7
17		5

S = 70

I		D SUB I
0		49645 45618 99762 02944 92641 67416 34942
1		70439 61687 66075 98029 25281 36674 498
2	-	49601 48623 15802 45403 62728 43923 7
3		91954 65396 04803 80746 49586 65
4		47576 09663 00779 01682 52778 88
5	-	69252 48317 43018 16555 45476
6		53212 68153 54428 88249 820
7	-	91379 70021 74880 73075
8	-	39053 64705 27594 1398
9		66429 83096 49653 15
10	-	59774 59761 24495
11		23475 26439 757
12		23725 36896 1
13	-	58700 0837
14		65238 96
15	-	41225
16	-	12
17		4

APPENDIX B

VALUES OF $\Pi_{s+1}(s)$

TABLE 4—$\Pi_{s+1}(s)$ TO 33 DECIMAL PLACES FOR $s = 1(1)70$

s	$\Pi_{s+1}(s)$							
1	0.	18927	51878	82093	32118	03671	35892	330
2	0.	28121	25252	33780	21211	01257	13387	264
3	0.	33107	18207	57297	57403	96445	04243	578
4	0.	36234	77109	30150	41806	24897	98150	113
5	0.	38381	35230	66598	99050	60316	80043	921
6	0.	39946	78498	64246	19181	32889	13640	946
7	0.	41139	37882	53067	68776	44516	94770	425
8	0.	42078	39223	27607	49424	80917	28837	913
9	0.	42837	06049	14723	28266	78893	32891	149
10	0.	43462	86186	56620	59289	61348	53475	084
11	0.	43987	93542	19413	10305	58450	13253	220
12	0.	44434	81780	57968	30769	29839	53993	445
13	0.	44819	78298	88886	96046	33589	61570	798
14	0.	45154	87794	63734	23189	51342	86097	413
15	0.	45449	21231	00126	13265	38379	62923	535
16	0.	45709	80274	88442	11242	59120	41783	262
17	0.	45942	14175	81479	01975	16278	75007	598
18	0.	46150	59042	79618	29366	20249	99240	751
19	0.	46338	65565	13054	01880	86860	44928	381
20	0.	46509	18960	03773	84625	06853	99018	443
21	0.	46664	53577	45605	86994	12593	13404	848
22	0.	46806	63760	81585	38061	44761	73415	886
23	0.	46937	12038	05492	33636	76180	48188	668
24	0.	47057	35378	63878	17896	97589	84742	077
25	0.	47168	50029	32419	66515	27559	25603	593
26	0.	47271	55291	67563	97837	78613	41673	717
27	0.	47367	36502	08385	52941	09116	90768	922
28	0.	47456	67404	08816	47666	49270	58888	022
29	0.	47540	12052	86421	56212	83451	19715	499
30	0.	47618	26356	11387	92503	91055	99968	335
31	0.	47691	59329	86460	13692	67473	83086	414
32	0.	47760	54128	88845	91697	19203	46973	181
33	0.	47825	48897	57110	72096	24500	39044	602
34	0.	47886	77476	70871	43722	72844	01478	426
35	0.	47944	69993	91774	07261	54744	48704	332
36	0.	47999	53359	42388	34689	01441	77717	049
37	0.	48051	51684	46461	18660	77452	11110	868
38	0.	48100	86636	04256	21229	19929	37206	665
39	0.	48147	77739	04861	66515	10377	15802	150
40	0.	48192	42634	64571	57171	69157	55298	186
41	0.	48234	97302	12808	35693	09826	06139	781
42	0.	48275	56250	14162	47536	06970	47051	474
43	0.	48314	32682	09154	01245	76918	66207	700
44	0.	48351	38639	71345	30964	37588	33777	672
45	0.	48386	85128	09929	90637	21073	71037	023
46	0.	48420	82224	91418	05908	29144	30722	574
47	0.	48453	39176	08848	49191	41823	57802	075
48	0.	48484	64479	89993	03458	33150	62033	238
49	0.	48514	65961	05653	84964	75672	58702	252
50	0.	48543	50836	14099	49566	43636	27540	305
51	0.	48571	25771	56932	67823	84126	40001	115
52	0.	48597	96935	04424	43113	64502	70622	856
53	0.	48623	70041	33944	51798	55257	61952	156
54	0.	48648	50393	13051	35804	28421	57737	357
55	0.	48672	42917	48662	39469	63201	53719	355
56	0.	48695	52198	55172	54534	96512	06715	972
57	0.	48717	82506	87152	26328	15260	09357	575
58	0.	48739	37825	76116	20313	32358	32871	349
59	0.	48760	21875	05627	24450	43893	17073	235
60	0.	48780	38132	54539	88146	12991	70518	604
61	0.	48799	89853	34369	13594	81214	77751	465
62	0.	48818	80087	43494	86428	50419	24013	489
63	0.	48837	11695	58092	56101	60483	27847	218
64	0.	48854	87363	77250	61249	67474	94690	823
65	0.	48872	09616	37632	07763	94802	14219	180
66	0.	48888	80828	11217	65400	90200	98587	768
67	0.	48905	03234	98084	54943	44213	32187	075
68	0.	48920	78944	24799	03848	79708	25702	805
69	0.	48936	09943	57799	44042	33997	18441	727
70	0.	48950	98109	40096	49161	53941	71702	657

APPENDIX C

VALUES OF $\Omega_s(s)$ AND OF $\Omega_{s+1}(s)$

TABLE 5—$\Omega_s(s)$ TO 33 DECIMAL PLACES FOR $s = 1(1)70$

s	$\Omega_s(s)$							
1	0.	37855	03757	64186	64236	07342	71784	661
2	0.	42181	87878	50670	31816	51885	70080	896
3	0.	44142	90943	43063	43205	28593	38991	437
4	0.	45293	46386	62688	02257	81122	47687	641
5	0.	46057	62276	79918	78860	72380	16052	705
6	0.	46604	58248	41620	55711	55037	32581	103
7	0.	47016	43294	32077	35744	50876	51166	200
8	0.	47338	19126	18558	43102	91031	94942	652
9	0.	47596	73387	94136	98074	20992	58767	943
10	0.	47809	14805	22282	65218	57483	38822	593
11	0.	47986	83864	21177	93060	63763	78094	421
12	0.	48137	71928	96132	33333	40659	50159	566
13	0.	48267	45860	34185	95742	20788	81691	629
14	0.	48380	22637	11143	81988	76438	77961	514
15	0.	48479	15979	73467	87483	07604	93785	104
16	0.	48566	66542	06469	74445	25315	44394	716
17	0.	48644	62068	50977	78561	93706	91184	515
18	0.	48714	51211	84041	53219	88041	65865	237
19	0.	48777	53226	45320	01979	86168	89398	296
20	0.	48834	64908	03962	53856	32196	68969	365
21	0.	48886	65652	57301	38755	75097	56900	317
22	0.	48934	21204	48930	17064	24069	08571	154
23	0.	48977	86474	49209	39447	05579	63327	306
24	0.	49018	07686	08206	43642	68322	75772	997
25	0.	49055	24030	49716	45175	88661	62627	736
26	0.	49089	68956	74008	74677	70098	54815	013
27	0.	49121	71187	34622	03050	02047	16352	957
28	0.	49151	55525	66274	20797	43887	39562	594
29	0.	49179	43502	96298	16771	89777	10050	517
30	0.	49205	53901	31767	52254	04091	19967	279
31	0.	49230	03179	21507	23811	79327	82540	814
32	0.	49253	05820	41622	35187	72928	57816	093
33	0.	49274	74621	73992	86402	19182	22045	953
34	0.	49295	20931	90602	95008	69104	13286	615
35	0.	49314	54850	88681	90326	16308	61524	456
36	0.	49332	85397	18565	80097	04259	60431	411
37	0.	49350	20648	90960	13759	71437	30330	081
38	0.	49366	67863	30684	00735	23085	40817	367
39	0.	49382	33578	51140	16938	56797	08515	026
40	0.	49397	23700	51185	86100	98386	49180	640
41	0.	49411	43577	78974	41441	71041	33118	800
42	0.	49424	98065	62118	72477	40469	76743	176
43	0.	49437	91581	67506	43135	20567	93328	810
44	0.	49450	28154	25239	52122	65715	34545	346
45	0.	49462	11464	27928	34873	59319	79282	291
46	0.	49473	44881	97753	23428	03690	92260	021
47	0.	49484	31498	98398	45982	72500	67542	544
48	0.	49494	74156	56451	22280	38007	92492	263
49	0.	49504	75470	46585	56086	48645	49696	176
50	0.	49514	37852	86381	48557	76509	00091	111
51	0.	49523	63531	79617	63271	36756	32942	313
52	0.	49532	54568	41047	97788	90743	14288	680
53	0.	49541	12872	30811	39568	33658	70668	234
54	0.	49549	40215	22552	30911	77096	05102	864
55	0.	49557	38243	25910	80187	26168	83786	980
56	0.	49565	08487	81157	76937	37521	21121	614
57	0.	49572	52375	41312	82930	40089	21802	445
58	0.	49579	71236	55014	75835	96709	33438	096
59	0.	49586	66313	61654	82491	97179	49566	001
60	0.	49593	38768	08782	21281	89874	90027	247
61	0.	49599	89687	00506	33489	80906	82304	768
62	0.	49606	20088	84518	97499	93167	93755	642
63	0.	49612	30928	84411	49055	59856	02892	412
64	0.	49618	23103	83145	15331	70091	74295	367
65	0.	49623	97456	62826	41729	54722	17514	860
66	0.	49629	54780	05327	01240	30961	60687	583
67	0.	49634	95820	57757	45315	73231	43115	241
68	0.	49640	21281	66340	20081	86762	79021	964
69	0.	49645	31826	81825	51927	01156	56390	158
70	0.	49650	28082	39240	72720	98998	02726	981

TABLE 6–$\Omega_{s+1}(s)$ TO 33 DECIMAL PLACES FOR $s = 1(1)70$

s	$\Omega_{s+1}(s)$							
1	0.	32831	10192	21786	48358	37319	54362	913
2	0.	39738	93029	17823	59171	37773	62032	204
3	0.	42711	79017	89185	94444	23749	80529	562
4	0.	44356	03447	62812	42491	94003	53226	658
5	0.	45396	91176	62568	92727	14273	16197	742
6	0.	46114	21975	29594	40596	77057	54494	642
7	0.	46638	26028	20938	97302	64716	89538	318
8	0.	47037	76779	37254	64596	68230	10259	464
9	0.	47352	38104	17725	26505	91914	30579	098
10	0.	47606	54704	32718	39072	35206	81971	967
11	0.	47816	15542	66257	33622	68550	47858	570
12	0.	47991	97838	99778	75677	39533	25193	061
13	0.	48141	57601	88702	04263	53453	74255	921
14	0.	48270	40988	18879	67510	22116	68149	137
15	0.	48382	52363	17696	21425	52780	75400	410
16	0.	48480	97497	48567	96695	72051	55266	815
17	0.	48568	11884	61598	61337	60970	23942	948
18	0.	48645	79831	62151	87526	27651	45107	715
19	0.	48715	47648	67813	90737	27925	33725	369
20	0.	48778	32960	81883	39373	94471	81255	988
21	0.	48835	31409	54568	45518	24475	97789	256
22	0.	48887	21559	64247	75745	90204	92394	972
23	0.	48934	68547	93890	03658	31230	64982	933
24	0.	48978	26834	87328	34563	29097	11182	629
25	0.	49018	42306	08563	70352	82098	99452	995
26	0.	49055	53896	37119	69437	60377	61411	059
27	0.	49089	94858	01689	61942	34674	53109	878
28	0.	49121	93761	17374	74425	79896	15554	971
29	0.	49151	75290	06798	06106	12844	47716	575
30	0.	49179	60882	06366	87736	55472	52924	354
31	0.	49205	69244	61157	17741	11496	69960	855
32	0.	49230	16776	46804	27948	89554	88599	744
33	0.	49253	17913	24803	62862	09409	95666	890
34	0.	49274	85412	70992	70905	02705	54366	353
35	0.	49295	30591	68992	70905	58356	67401	450
36	0.	49314	63523	98443	03347	27528	98496	349
37	0.	49332	93206	48955	26040	96156	42492	751
38	0.	49350	27699	38428	55100	64882	19083	261
39	0.	49366	74244	96872	21036	93523	38290	892
40	0.	49382	39368	85561	54508	07946	70446	855
41	0.	49397	28966	49887	03116	11072	92890	680
42	0.	49411	48377	47961	33755	55871	64881	566
43	0.	49425	02449	52428	13591	46869	04019	808
44	0.	49437	95593	87341	19908	97062	43581	998
45	0.	49450	31833	33459	57188	72082	61111	928
46	0.	49462	14844	12314	50249	51050	82129	201
47	0.	49473	47992	40779	05586	33237	32255	050
48	0.	49484	34366	32709	91039	09811	96977	456
49	0.	49494	76804	11831	64677	15976	70341	108
50	0.	49504	77918	89848	78624	28509	93969	865
51	0.	49514	40120	55369	14106	29085	31045	161
52	0.	49523	65635	12263	35656	31244	40110	418
53	0.	49532	56522	00299	61828	56071	48920	330
54	0.	49541	14689	26063	78186	73822	19095	414
55	0.	49549	41907	28130	87808	39973	69108	309
56	0.	49557	39820	97054	97091	38923	39963	872
57	0.	49565	09960	67878	27659	55209	06642	780
58	0.	49572	53752	00436	03711	44190	12277	687
59	0.	49579	72524	60676	53144	48415	79140	247
60	0.	49586	67520	14464	99217	11238	28968	917
61	0.	49593	39899	43846	15819	65099	26178	188
62	0.	49599	90748	94461	61971	05846	51392	283
63	0.	49606	21086	61721	01673	95654	60572	847
64	0.	49612	31867	22382	35917	07268	87640	302
65	0.	49618	23987	17382	80521	29046	46161	145
66	0.	49623	98288	91057	61076	25702	62884	651
67	0.	49629	55564	91275	27955	45020	86465	902
68	0.	49634	96561	34487	47036	68843	39642	663
69	0.	49640	21981	39231	39658	49052	77533	691
70	0.	49645	32488	31220	53434	57958	91279	656

APPENDIX D

VALUES OF THE SINE AND COSINE INTEGRALS

TABLE 7–$si(x)$ TO 33 DECIMAL PLACES FOR $x = 1(1)70$

x				$si(x)$			
1	−0.	62471	32564	27713	60428	99683	778
2	0.	03461	66500	07798	22934	53984	566
3	0.	27785	62012	04571	63716	64085	595
4	0.	18740	68121	54156	43887	42376	117
5	−0.	02086	50818	50222	48195	69132	909
6	−0.	14610	87755	14390	08346	22906	637
7	−0.	11619	97125	46803	02861	65531	978
8	0.	00339	04949	12045	43285	16497	596
9	0.	09424	37490	34705	87587	53317	363
10	0.	08755	12674	23977	43009	96501	877
11	0.	00751	04801	50830	79648	79540	535
12	−0.	06582	50852	68523	24870	41731	595
13	−0.	07143	46039	32072	05500	37149	094
14	−0.	01458	52767	17231	56552	76897	988
15	0.	04739	81169	13472	11989	26669	398
16	0.	06050	59414	75136	26691	52816	252
17	0.	01934	00890	75804	50320	25257	183
18	−0.	03418	82299	33711	15687	01477	978
19	−0.	05216	62950	25532	68765	96871	414
20	−0.	02255	46257	51456	77906	76783	495
21	0.	02409	46412	74762	70221	22667	843
22	0.	04528	74097	99469	92388	31093	355
23	0.	02466	31055	18260	21956	78927	114
24	−0.	01605	76350	71977	51917	32659	077
25	−0.	03931	37757	94935	29660	01384	986
26	−0.	02592	74638	08558	06134	35844	313
27	0.	00948	93571	73776	11125	41737	632
28	0.	03394	94114	95475	33348	92680	551
29	0.	02651	81882	49224	38162	86729	478
30	−0.	00403	97867	64545	50824	75903	827
31	−0.	02902	92895	01107	14690	72349	174
32	−0.	02655	45497	35755	10385	70639	013
33	−0.	00051	16286	26210	71451	80734	751
34	0.	02445	98583	87572	00573	57929	860
35	0.	02612	58777	13409	00612	20695	532
36	0.	00431	08827	61880	38152	63169	366
37	−0.	02018	88557	69319	97574	66044	853
38	−0.	02530	33895	59197	87866	94305	609
39	−0.	00745	92690	98525	96467	75278	911
40	0.	01618	87925	59887	88754	43442	704
41	0.	02414	70246	47298	39350	70721	373
42	0.	01003	05569	20119	62773	59211	237
43	−0.	01244	73301	65561	89987	20906	100
44	−0.	02270	94954	70314	79949	57945	025
45	−0.	01208	13258	98483	77281	29753	503
46	0.	00896	13635	21708	92742	99872	906
47	0.	02103	87170	20935	61391	13018	596
48	0.	01365	74061	74167	95711	79077	921
49	−0.	00573	32100	58625	46653	53640	049
50	−0.	01917	92543	08960	72450	33360	968
51	−0.	01479	79959	62744	98483	90623	664
52	0.	00276	80565	70212	01420	85714	223
53	0.	01717	31564	90128	80314	66984	921
54	0.	01553	83360	67068	46608	93399	618
55	−0.	00007	21933	96744	84196	44242	888
56	−0.	01506	03985	88975	40875	74393	911
57	−0.	01591	14541	69725	34659	16171	650
58	−0.	00234	82357	00524	98962	89698	116
59	0.	01287	90685	01941	72474	33178	752
60	0.	01594	92894	65050	79309	51184	407
61	0.	00448	81302	69302	71332	67505	525
62	−0.	01066	53882	57229	90227	73770	974
63	−0.	01568	33923	47207	34735	87181	263
64	−0.	00634	40765	82866	08706	82213	305
65	0.	00845	36290	83705	75085	62118	791
66	0.	01514	53253	85134	29094	97632	313
67	0.	00791	48013	70666	04146	39882	175
68	−0.	00627	59615	01426	39967	09696	243
69	−0.	01436	68501	65808	57961	57054	532
70	−0.	00920	14776	16890	51370	83394	869

TABLE 8–$Si(x)$ TO 33 DECIMAL PLACES FOR $x = 1(1)70$

x				$Si(x)$			
1	0.	94608	30703	67183	01494	13533	139
2	1.	60541	29768	02694	84857	67201	482
3	1.	84865	25279	99468	25639	77302	511
4	1.	75820	31389	49053	05810	55593	033
5	1.	54993	12449	44674	13727	44084	007
6	1.	42468	75512	80506	53576	90310	279
7	1.	45459	66142	48093	59061	47684	938
8	1.	57418	68217	06942	05208	29714	512
9	1.	66504	00758	29602	49510	66534	279
10	1.	65834	75942	18874	04933	09718	794
11	1.	57830	68069	45727	41571	92757	452
12	1.	50497	12415	26373	37052	71485	321
13	1.	49936	17228	62824	56422	76067	823
14	1.	55621	10500	77665	05370	36318	928
15	1.	61819	44437	08368	73912	39886	314
16	1.	63130	22682	70032	88614	66033	169
17	1.	59013	64158	70701	12243	38474	100
18	1.	53660	80968	61185	46236	11738	939
19	1.	51863	00317	69363	93157	16345	502
20	1.	54824	17010	43439	84016	36433	422
21	1.	59489	09680	69659	32144	35884	759
22	1.	61608	37365	94366	54311	44310	272
23	1.	59545	94323	13156	83879	92144	030
24	1.	55473	86917	22919	10005	80557	839
25	1.	53148	25509	99961	32263	11831	930
26	1.	54486	88629	86338	55788	77372	603
27	1.	58028	56839	68672	73048	54954	548
28	1.	60474	57382	90371	95272	05897	467
29	1.	59731	45150	44121	00085	99946	394
30	1.	56675	65400	30351	11098	37313	090
31	1.	54176	70372	93789	47232	40867	743
32	1.	54424	17770	59141	51537	42577	904
33	1.	57028	46981	68685	90471	32482	165
34	1.	59525	61851	82468	62496	71146	776
35	1.	59692	22045	08305	62535	33912	448
36	1.	57510	72095	56777	00075	76386	282
37	1.	55060	74710	25576	64348	47172	063
38	1.	54549	29372	35698	74056	18911	308
39	1.	56333	70576	96370	65455	37938	006
40	1.	58698	51193	54784	50677	56659	620
41	1.	59494	33514	42195	01273	83938	289
42	1.	58082	68837	15016	24696	72428	153
43	1.	55834	89966	29334	71935	92310	816
44	1.	54808	68313	24581	81973	55271	892
45	1.	55871	50008	96412	84641	83463	414
46	1.	57975	76903	16605	54666	13089	822
47	1.	59183	50438	15832	23314	26235	512
48	1.	58445	37329	69064	57634	92294	838
49	1.	56506	31167	36271	15269	59576	867
50	1.	55161	70724	85935	89472	79855	949
51	1.	55599	83308	32151	63439	22593	253
52	1.	57356	43833	65108	63343	98931	139
53	1.	58796	94832	85025	42237	80201	837
54	1.	58633	46628	61965	08532	06616	534
55	1.	57072	41333	98151	77726	68974	029
56	1.	55573	59282	05921	21047	38823	005
57	1.	55488	48726	25171	27263	97045	266
58	1.	56844	80910	94371	62960	23518	800
59	1.	58367	53952	96838	34397	46395	668
60	1.	58674	56162	59947	41232	64401	323
61	1.	57528	44570	64199	33255	80722	442
62	1.	56013	09385	37666	71695	39445	942
63	1.	55511	29344	47689	27187	26035	653
64	1.	56445	22502	12030	53216	31003	611
65	1.	57924	99558	78602	37008	75335	708
66	1.	58594	16521	80030	91018	10849	229
67	1.	57871	11281	65562	66069	53099	092
68	1.	56452	03652	93470	21956	03520	674
69	1.	55642	94766	29088	03961	56162	384
70	1.	56159	48491	78006	10552	29822	047

TABLE 9–$Ci(x)$ TO 33 DECIMAL PLACES FOR $x = 1(1)70$

x	$Ci(x)$						
1	0.	33740	39229	00968	13466	26462	039
2	0.	42298	08287	74864	99569	85651	532
3	0.	11962	97860	08000	32762	64722	812
4	–0.	14098	16978	86930	41163	91448	987
5	–0.	19002	97496	56643	87861	84589	001
6	–0.	06805	72438	93247	12620	41683	048
7	0.	07669	52784	82184	51838	29157	630
8	0.	12243	38825	32009	55729	22959	583
9	0.	05534	75313	33133	60708	56416	484
10	–0.	04545	64330	04455	37263	45328	300
11	–0.	08956	31354	95479	97948	24676	305
12	–0.	04978	00068	84113	67559	59212	087
13	0.	02676	41255	64034	55503	67498	668
14	0.	06939	63559	27584	54727	43832	682
15	0.	04627	86776	74360	43960	43108	309
16	–0.	01420	01901	20190	02239	87727	863
17	–0.	05524	26822	60813	85052	73400	995
18	–0.	04347	51029	99501	00478	34411	492
19	0.	00515	03710	08426	12856	98550	925
20	0.	04441	98208	45353	31653	97687	170
21	0.	04089	05001	54692	36971	03028	992
22	0.	00164	06919	15737	74972	66809	806
23	–0.	03565	98603	82261	68292	95312	957
24	–0.	03833	30155	51247	14767	84692	866
25	–0.	00684	85971	79702	59091	89387	780
26	0.	02829	51510	31757	13190	84211	300
27	0.	03572	32177	17395	86892	52412	198
28	0.	01086	95343	08760	86639	85990	747
29	–0.	02194	69729	74023	04438	17170	542
30	–0.	03303	24172	82071	14377	92264	410
31	–0.	01395	27170	88765	38111	27878	536
32	0.	01638	88233	76158	67976	52215	894
33	0.	03025	74341	71747	31064	64190	945
34	0.	01626	49164	37355	76698	16563	519
35	–0.	01147	98563	55303	12949	95524	260
36	–0.	02740	89958	44977	44812	19314	772
37	–0.	01792	44197	10374	87873	38841	573
38	0.	00712	97618	01971	37971	35513	765
39	0.	02450	58334	17995	32807	86490	871
40	0.	01902	00078	96208	76696	19812	034
41	–0.	00327	89946	01272	07220	24609	597
42	–0.	02157	13793	86110	01039	32740	436
43	–0.	01962	20686	19766	02916	38534	024
44	–0.	00011	29602	15923	15149	71254	859
45	0.	01863	17437	03556	53210	28213	528
46	0.	01978	93378	69101	64954	65696	877
47	0.	00307	48622	02395	11907	40950	517
48	–0.	01571	43603	39703	32720	89430	021
49	–0.	01957	30908	02871	69756	81183	920
50	–0.	00562	83863	24116	30544	01858	955
51	0.	01284	70085	40631	13274	41575	407
52	0.	01901	97802	86359	47487	24315	938
53	0.	00779	12035	06269	68417	72858	007
54	–0.	01005	70583	27335	24933	98636	613
55	–0.	01817	26961	21802	92912	23331	524
56	–0.	00957	90490	58468	44996	70886	149
57	0.	00737	08611	60588	57036	19831	016
58	0.	01707	29871	71210	07125	73603	436
59	0.	01100	71216	77743	47251	35276	728
60	–0.	00481	32433	77443	21528	88723	439
61	–0.	01576	02584	39320	85240	81906	190
62	–0.	01209	10233	69152	33626	71637	466
63	0.	00240	70795	07390	65877	50053	950
64	0.	01427	28792	13325	48725	35777	047
65	0.	01284	73736	99511	79921	86448	815
66	–0.	00017	29331	94092	48378	73381	330
67	–0	01264	80926	03150	42962	90900	197
68	–0.	01329	41857	91036	14117	61348	849
69	–0.	00187	12407	20841	47510	74718	745
70	0.	01092	19884	73464	97703	78113	544

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13. ABSTRACT

The theory of the converging factors for the sine and cosine integrals is developed and then applied to the calculation on a CDC 6700 system of tables of these factors and their reduced derivatives to about 35 decimal places. The factors are then used in conjunction with appropriately truncated asymptotic series to produce tables of si(x), Si(x), and Ci(x) to 28 decimals by a computer program, for successive integer values of x ranging from 1 to 70, inclusive. Interpolation in the tables by Taylor series is illustrated by numerical examples, which include a new, extended calculation of the Gibbs constant.

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Sine and Cosine Integrals Asymptotic Expansions Converging Factors Gibbs Constant						